HOMEWORK ASSIGNMENT NO 14 — MATHEMATICAL FINANCE I

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Due date of this homework: 5 February 2020, 12 p.m. (noon), MA 141

The total score of this homework is 16 points.

Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$ and write $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

1. A "universality" result for European call options (4 points)

We consider an arbitrage-free continuous-time financial market (say, the Black-Scholes one) with risky asset S^1 and time horizon $T > 0$. Within this framework, consider a European option C with payoff $C(T) = g(S^1(T))$ where $g \in C^2(\mathbb{R})$ with $g \geq 0$ and $g(0) = g'(0) = 0$. Then, assuming that, for any $t \in [0, T]$, there is a unique time-t no-arbitrage price $C(t, K)$ of the European call option on S^1 with maturity T and strike $K > 0$,[1] show that the no-arbitrage price of C at time $t \in [0, T]$ has to be given by

$$C(t) = \int_0^\infty g''(K) C(t, K)\, dK,$$

where $C(t, K)$ is the time-t no-arbitrage price .

Hint: Argue why it suffices to show the result for $t = T$. Refer to the comparision principle discussed earlier during the course.

2. A non-admissible continuous-time strategy with seemingly risk-free profits (in total: 12 points)

We consider a Black-Scholes model as introduced in the lecture with time horizon $T > 0$ and $r = 0$. In the following, we will construct a non-admissible self-financing trading strategy φ with

$$X^\varphi(0) = 0, \qquad X^\varphi(T) \geq 1. \tag{1}$$

2.1. **A profitable time of liquidation (4 point).**

Let $(t_n)_{n \in \mathbb{N}} \in [0, 1)^{\mathbb{N}}$ be a strictly increasing sequence starting in $t_0 = 0$ and converging to T. For $n \in \mathbb{N}^*$, set $\varepsilon_{n-1} = \mathrm{e}^{\sqrt{t_n - t_{n-1}}} - 1$, and consider the event

$$F_n = \{S^1(t_n) > (1 + \varepsilon_{n-1}) S^1(t_{n-1})\}.$$

Show that $\limsup_{n \to \infty} F_n = \bigcap_{n \in \mathbb{N}^*} \bigcup_{k \geq n} F_k$ has probability measure one. Deduce that the formula

$$\tau = \sum_{n \in \mathbb{N}^*} t_n \mathbf{1}_{G_n},$$

where $G_n = F_n \setminus \left(\bigcup_{k=1}^{n-1} F_k \right)$ for $n \in \mathbb{N}^*$, defines a stopping time with values in $[0, T)$.

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[1] We assume that it exists under an appropriate notion of arbitrage (see definition 5.31 in the lecture notes). In fact, in the above framework it does, and it is given by means of the Black-Scholes formula.

2.2. **Seemingly risk-free profits (4 points).** Define recursively $X(t_0) = 0$, and, for $n \in \mathbb{N}^*$ and $t \in (t_{n-1}, t_n]$,

$$h_n = \mathbf{1}_{\tau > t_{n-1}} \frac{|1 - X(t_{n-1})|}{\varepsilon_{n-1} S^1(t_{n-1})},$$
$$X(t) = X(t_{n-1}) + h_{n-1}(S^1(t) - S^1(t_{n-1})).$$

Finally, put $X(T) = X(\tau)$. Show that the formula

$$\varphi_1 = \sum_{n=1}^{\infty} h_n \mathbf{1}_{(t_{n-1}, t_n]}$$

defines a self-financing trading strategy $\varphi = (\varphi_0, \varphi_1)$ with initial wealth 0 and wealth process X. Then, deduce that φ satisfies (1).

2.3. **... an arbitrage strategy? (4 points).** Show that φ is not admissible (cf. definition 5.31 in the lecture notes).

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[2]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".