HOMEWORK ASSIGNMENT NO 13 — MATHEMATICAL FINANCE I

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Due date of this homework: 29 January 2020, 12 p.m. (noon), MA 141

The total score of this homework is 16 points (exercises nos one to four).
Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Moments of $\mathcal{N}(0,t)$ and Brownian motion (4 points)

Let be W a Wiener process with time horizon $T > 0$ and values in $\mathbb{R}$ and write $\mu_k(t) = \mathbb{E}[W(t)^k]$ for $k \in \mathbb{N}$ and $t \in [0,T]$. Using Itô's formula, prove the following for any $t \in [0,T]$:

(1) $\mu_k(t) = \frac{1}{2}k(k-1)\int_0^t \mu_{k-2}(t)\ dt$ for all $k \geq 2$;
(2) $\mu_4(t) = 3t^2$.

2. Applying Itô's formula (4 points)

Let be $f \in C^1([0,\infty))$ and W an $\mathbb{R}$-valued Wiener process with time horizon $T > 0$. Show the following statements:

(1) $\mathbb{E}\Big[\Big(\int_0^T f'(s)W(s)\ ds\Big)^2\Big] = f(T)^2 T - 2f(T)\int_0^T f(s)\ ds + \int_0^T f(s)^2\ ds$;
(2) $\mathbb{E}\big[\big|\int_0^T W(s)\ ds\big|^2\big] = \frac{1}{3}T^3$.

3. A stochastic differential equation (4 points)

Let W be an $\mathbb{R}$-valued Wiener process with time horizon $T > 0$, $x \in \mathbb{R}$, $\alpha, \beta, \sigma > 0$. Define the process X by

$$X(t) = x\,\mathrm{e}^{-\beta t} + \frac{\alpha}{\beta}(1 - \mathrm{e}^{-\beta t}) + \sigma \int_0^t \mathrm{e}^{-\beta(t-s)}\ dW(s),$$

$t \in [0,T]$[1]. Show that X satisfies the equation

$$dX(t) = (\alpha - \beta X(t))\ dt + \sigma\ dW(t), \tag{1}$$

for $t \in [0,T]$.

4. Harmonic functions of Brownian motion are local martingales (4 points)

Let be W a Brownian motion with values in $\mathbb{R}$ and with time horizon $T > 0$ and $f \in C^2(\mathbb{R})$ a superharmonic function, i.e. $\Delta f \leq 0$ with Δ denoting the second derivative.

Show that the process $t \mapsto f(W(t))$ is a local supermartingale[2].

Date: January 22, 2020.

[1] The last integral is defined as follows: Let H be the process defined by $H(s) = \mathrm{e}^{-\beta(t-s)}$, $s \in [0,T]$. Then $\int_0^t \mathrm{e}^{-\beta(t-s)}\ dW(s) = H \bullet W(t)$.

[2] ... i.e. there exists a localizing sequence of stopping times such that the respective sequence of stopped processes is a sequence of supermartingales (replace "martingale" with "supermartingale" in the definition on last week's homework assignment)

Remark: This statement holds true also in higher dimensions.

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[3]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".