HOMEWORK ASSIGNMENT NO 12 — MATHEMATICAL FINANCE I

PROFESSOR DR. C. BELAK (LECTURER),
E. RAPSCH, DR. D. BESSLICH (ASSISTANTS)

Due date of this homework: 22 January 2020, 12 p.m. (noon), MA 141

The total score of this homework is 16 points (exercises nos one to four).
Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Stochastic integration of deterministic functions (4 points)

Let be $f \in L^2([0,T])$ and W a Wiener process with time horizon $T > 0$. Show that:

(1) for $t \in [0,T]$, $f \bullet W(t)$ obeys the law $\mathcal{N}(0, \int_0^t |f(s)|^2 \, ds)$;

(2) the process given by $[0,T] \ni t \mapsto \exp\left(f \bullet W(t) - \frac{1}{2}\int_0^t |f(s)|^2 \, ds\right)$ is a martingale with respect to the natural filtration of W.

2. Bounded local martingales are martingales (4 points)

Definition. Given a filtered probability space with time $\mathcal{T} \subset [0,T]$, $T > 0$, a process X is called *local martingale* iff there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times taking values in $\mathcal{T} \cup \{\infty\}$ and with $\tau_n \longrightarrow \infty$ a.s. as $n \to \infty$, such that, for any $n \in \mathbb{N}$, X^{τ_n} is a martingale.

Show that any local martingale, which is bounded from below by a martingale, is a supermartingale.

Hint: You may first consider the special case in which the bounding martingale is a deterministic constant.

3. Existence of strict local martingales (4 points)

It follows immediately from the definitions that any martingale is a local martingale. However, the converse implication does not hold in general.

In order to see this, let W be a Wiener process with infinite time horizon[1], let be $\tau = \inf\{t > 0 \mid W(t) = 1\}$, and define the process M by $M(t) = W^\tau\left(\frac{t}{1-t}\right)$ for $t \in [0,1)$ and $M(1) = 1$. Show that M is a local martingale, but not a martingale.

4. There are no integrable strict local martingales in discrete time (4 points)

Show that, on a filtered (finite or infinite) probability space with time $\{0,\dots,T\} = [0,T] \cap \mathbb{N}$, $T \in \mathbb{N}$, any local martingale X with $\mathbb{E}[X(t)_-] < \infty$, $t = 0,\dots,T$, is a martingale.

Technische Universität Berlin, Fakultät II, Institut für Mathematik, Lehrstuhl für Stochastik und Quantitative Finanzmathematik, Strasse des 17. Juni 136, D-10623 Berlin

Email address: `profname@math.tu-berlin.de, assi1name@tu-berlin.de, assi2name@math.tu-berlin.de` [2]

Date: January 15, 2020.

[1] Such processes exist, cf. standard literature about stochastic processes and Brownian motion: Karatzas/Shreve, Revuz/Yor, for instance.

[2] The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".