HOMEWORK ASSIGNMENT NO. 11 — MATHEMATICAL FINANCE I

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Due date of this homework: 15 January 2020, 12 p.m. (noon), MA 141

The total score of this homework is 16 points (exercises one, two, three).
Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Blumenthal's zero-one law for Brownian motion (4 points)

Show that if W is a Wiener process on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, then $\mathfrak{F}^W(0+) = \bigcap_{t>0} \mathfrak{F}^W(t)$ is trivial in the following sense: $\mathbb{P}(F) \in \{0, 1\}$ for all $F \in \mathfrak{F}^W(0+)$.

Hint: You are allowed to apply Kolmogorov's zero-one law.

2. Predictable processes are adapted (4 points)

Let $(\Omega, \mathfrak{A}, \mathfrak{F})$ be a filtered measurable space with time $[0, T]$ for some $T > 0$. Show that any $\mathfrak{F}$-predictable process on it is adapted to $\mathfrak{F}$.

3. Continuous martingales of bounded variation (8 points)

Let us assume that X is a square-integrable martingale on a filtered probability space $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ with time $\mathcal{T} = [0, T]$ for some $T > 0$. Prove the following statements:

(1) For all $s, t \in \mathcal{T}$ with $s < t$, it holds that a.s.

$$\mathbb{E}_s[(X(t) - X(s))^2] = \mathbb{E}_s[X(t)^2 - X(s)^2].$$

(2) For any finite, increasing sequence $t_0, \ldots, t_n \in \mathcal{T}$, $n \in \mathbb{N}$, we have

$$\mathbb{E}[X(t_n)^2] = \mathbb{E}[X(t_0)^2] + \sum_{k=1}^{n} \mathbb{E}[(X(t_k) - X(t_{k-1}))^2].$$

Note that this implies that the mapping $t \mapsto \mathbb{E}[X(t)^2]$ is non-decreasing.

(3) Deduce that if X is a bounded continuous martingale of finite variation[1], then—for all $t \in [0, T]$—$\mathbb{P}[X(t) = X(0)] = 1$.

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Date: January 8, 2020.

[1]The *total variation* of a real valued function f defined on, say, $[0, T]$, is the function $V_f : [0, T] \longrightarrow [0, \infty]$ given by

$$V_f(t) = \sup \left\{ \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \;\middle|\; 0 \le t_0 \le \cdots \le t_n \le t, n \in \mathbb{N} \right\}$$

for $t \in [0, T]$. The *total variation* of a stochastic process Y is the stochastic process V_Y given by $V_Y(t, \omega) = V_{Y(.,\omega)}(t)$. Finally, Y is said to be of *finite variation* iff its total variation process is pathwise bounded.

[2]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".