HOMEWORK ASSIGNMENT NO. 9 — MATHEMATICAL FINANCE I

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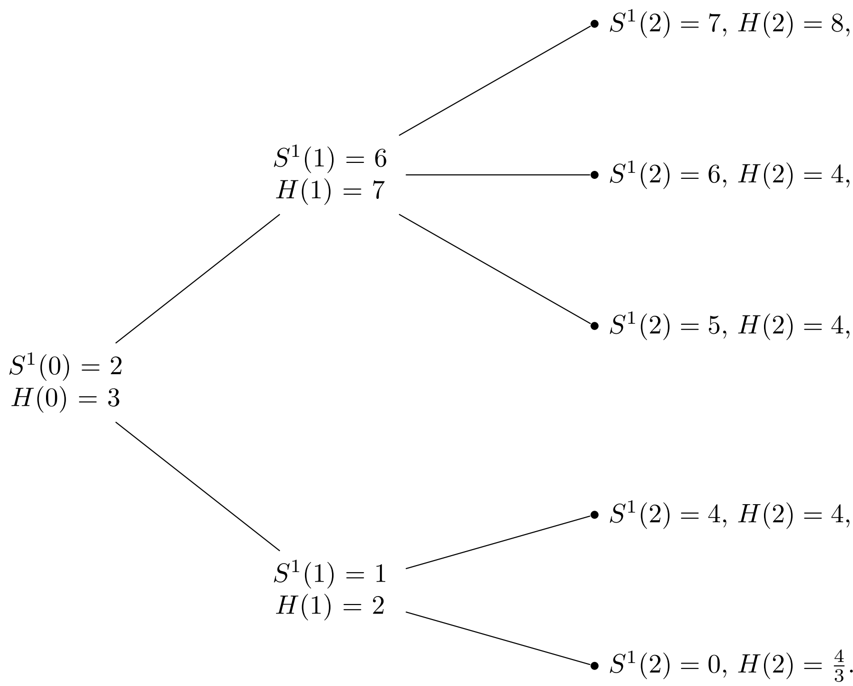
Return of this homework: 18 December 2019, 12 p.m. (noon), MA 141

The total score of this homework is 16 points (exercises one, two, three).
Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. A Multi-Period Market (6 points)

We consider a finite financial market model S with time horizon $T = 2$ and two assets, i.e. $S = (S^0, S^1)$, such that S^0 is deterministic and constant. The process S^1 and another process H, also defined on the underlying probability space, is given by the following diagramme:



The underlying filtration $\mathfrak{F}$ is generated by S^1.

(i) Every probability measure $\mathbb{P}'$ in this model can be entirely characterised by the numbers

$$\begin{aligned} &p'_1 = \mathbb{P}'(S_1 = 6), && p'_2 = \mathbb{P}'(S_2 = 7 | S_1 = 6), \\ &p'_3 = \mathbb{P}'(S_2 = 6 | S_1 = 6), && p'_4 = \mathbb{P}'(S_2 = 4 | S_1 = 1). \end{aligned} \tag{1}$$

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Assume that the underlying (physical) probability measure $\mathbb{P}$ satisfies $\mathbb{P}^{-1}(\{0\}) = \{\emptyset\}$. Describe the set $\mathcal{P}$ of $\mathbb{P}$-equivalent martingale measures viewed as a subset of $\mathbb{R}^4$, i.e.

$$\tilde{\mathcal{P}} = \left\{ p' = (p'_1, p'_2, p'_3, p'_4) \in \mathbb{R}^4 \mid \exists \mathbb{P}' \in \mathcal{P} : \ \mathbb{P}' \text{ satisfies (1) for } p' \right\}.$$

(ii) Calculate the superhedging process U for H.

(iii) Find all possible choices of an adapted increasing process C with $C(0) = 0$ and a predictable process φ_1 such that

$$U = U(0) + (\varphi_1 \bullet S^1) - C. \tag{2}$$

Why does there exist at least one pair (C, φ_1) with that property? Is there more than one pair (C, φ_1) with that property?

(iv) Calculate all possible superhedging strategies φ with initial capital $U(0)$ for the American option given by H. Does there exist a superhedging strategy for initial capital strictly smaller than $U(0)$? Please discuss this.

2. Continuous Functions (4 points)

2.1. **Preliminaries.** We know that the image of a compact set under continuous functions is compact, so in particular real-valued continuous functions are bounded and attain both their maximum and minimum. However, there are many cases, where only one of these two properties is needed and/or fulfilled, whence the interest in the following notion.

Definition. Let X be topological space, and let $h : X \longrightarrow \mathbb{R}$ be a map.

(1) h is called *upper semicontinuous* at a point $x_0 \in X$ if and only if, for any $y > f(x_0)$ there is a neighbourhood U of x_0 such that, for all $x \in U$, $f(x) \leq y$.

(2) h is called *upper semicontinuous* if and only if it is upper semicontinuous at every point $x_0 \in X$.

Remark. With the above notations, we have:

(1) h is upper semicontinuous if and only if, for any $y \in \mathbb{R}$, $h^{-1}((-\infty, y])$ is closed;

(2) h is upper semicontinuous if and only if the hypograph

$$\{(x, y) \in X \times \mathbb{R} \mid f(x) \geq y\}$$

is closed;

(3) in metric spaces, such as Euclidean space $X = \mathbb{R}^d$, for $d \in \mathbb{N}$, h is upper semicontinuous in x_0 if and only if

$$\limsup_{x \to x_0} f(x) \leq f(x_0);$$

(4) if X is compact and h is upper semicontinuous, then h is bounded from above and attains its supremum on X.

(5) In an analogous fashion—more precisely, by inverting the inequality signs in the above definition—one defines *lower semicontinuity*. Then, h is lower semicontinuous (at a point) if and only $-h$ is upper semicontinuous (at that point, respectively). Moreover, h is continuous (at a point) if and only if it is both upper and lower semicontinuous (at that point, respectively).

2.2. **Task.**

(1) Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a map, which is bounded from above, where $d \in \mathbb{N}$. Prove that f is upper semicontinuous if and only if there is a monotone decreasing sequence $(f_n) \in C_b^0(\mathbb{R}^d)$ of continuous, bounded maps whose pointwise limit is f, i.e.

$$\forall x \in \mathbb{R}^d : \ f_n(x) \searrow f(x), \text{ as } n \longrightarrow \infty.$$

(2) Construct a function $g \in C^0(\mathbb{R})$ which is continuous, but nowhere differentiable.

Note: In order to assure continuity (in the "if"-part of the first question) we would need *locally uniform* convergence (which is the same as uniform convergence on compact sets).

3. Stopping Processes and Filtrations (6 points)

Let $(\Omega, \mathfrak{A}, \mathfrak{F})$ be a *finite* filtered measurable space with time $\mathcal{T} = \{0, \ldots, T\}$, $T \in \mathbb{N}$, and τ a stopping time, X an adapted process on it. Prove or disprove the assertion that

$$(\mathfrak{F}^X)^\tau = \mathfrak{F}^{X^\tau}, \tag{3}$$

i.e. that the "stopping" the natural filtration associated to X yields the natural filtration of the stopped process of X. (You may compare exercise sheets 4 and 6 for the definitions.)

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[1]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".