HOMEWORK ASSIGNMENT NO. 8 — MATHEMATICAL FINANCE I

PROFESSOR DR. C. BELAK (LECTURER),
E. RAPSCH, DR. D. BESSLICH (ASSISTANTS)

Return of this homework: 11 December 2019, 12 p.m. (noon), MA 141

The total score of this homework is 16 points (exercises one, two, three). There is one *optional, but not compulsory* exercise which gives up to 8 points.

Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Stopping Times (4 points)

Let $(\Omega, \mathfrak{A}, \mathfrak{F})$ be a finite filtered measurable space with time $\mathcal{T} = \{0, \dots, T\}$, $T \in \mathbb{N}$, with $\mathfrak{F}(0) = \{\emptyset, \Omega\}$. Let σ, ρ be an $\mathfrak{F}$-stopping time with values in $\mathcal{T}$. Answer briefly whether—in general—the following random variables τ are $\mathfrak{F}$-stopping times or not:

(i) $\tau = (t-1) \vee 0$, for some $t \in \mathcal{T}$;
(ii) $\tau = (\sigma + \rho) \wedge T$;
(iii) $\tau = (\sigma - \rho) \vee 0$;
(iv) $\tau = \xi$ for some random variable ξ with values in $\mathcal{T}$;
(v) $\tau = \sigma \wedge \rho$;
(vi) $\tau = \sigma \vee \rho$;
(vii) $\tau = \mathbf{1}_A \sigma + \mathbf{1}_{A^C} T$, for some $A \in \mathfrak{F}(\sigma)$;
(viii) $\tau = \sigma \circ f$ for a measurable map $f : (\Omega, \mathfrak{A}) \longrightarrow (\Omega, \mathfrak{A})$.

2. Doob Decomposition (6 points)

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a finite probability space with time $\mathcal{T} = \{0, \dots, T\}$, $T \in \mathbb{N}$ such that there exists ξ alias a random variable with $\mathbb{E}[\xi] = 0$ and $\mathbb{E}[\xi^2] = 1$, and let W be a stochastic process with $W(0) = 0$ and such that the random variables $W(t) - W(t-1)$, for any $t = 1, \dots, T$, are i.i.d. copies of ξ. Let be $\mu, \sigma \in \mathbb{R}$ two real numbers and define the process X by $X(t) = \mu t + \sigma W(t)$, $t \in \mathcal{T}$. We equip our probability space with the filtration $\mathfrak{F}^X$ induced by X.

(i) Compute the Doob decomposition of X.
(ii) Compute the Doob decomposition of $Y = X^2 = X \cdot X$.
(iii) Compute the Doob decomposition of Z defined by $Z(t) = \exp\Big(X(t) - \frac{\sigma^2}{2} t\Big)$.
(iv) Give—for each of the three cases—the necessary and sufficient condition upon μ, σ under which X, Y, Z is a martingale, respectively.

3. A European Claim as an American One (6 points)

For $T = 2$, we consider the finite financial market given by the T-period CRR model with $r = 0$ and $(1+u)(1+d) = 1$ and let $\hat{\xi} = (S^1(T) - K)_+$ be a European call option on it, written on S^1, with maturity T and strike $K = S^1(0) = s$. Let ξ be the process defined by $\xi(t) = \mathbf{1}_{t=T} \hat{\xi}$, $t = 0, \dots, T$.

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Compute the Snell envelope and the time-0 superhedging price of ξ. Then, compare your result with the risk-neutral and the superhedging price of the European option $\hat{\xi}$ and discuss your findings.

4. Superreplicating American Contingent Claims in Incomplete Markets (8 bonus points)

This exercise is **not compulsory**, but you can achieve *8 bonus points.*

We consider a (possibly incomplete) arbitrage-free financial market model with asset structure $S = (S^0, S^1)$ on some finite filtered probability space $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ with $\mathbb{P}^{-1}(\{0\}) = \{\emptyset\}$ and time $\mathcal{T} = \{0, \dots, T\}$, $T \in \mathbb{N}$. Let be ξ an American contingent claim on it. By $\hat{P}_{\mathrm{a}}(0)$ we denote the time-0 superhedging price of ξ, by $\mathcal{P}$ the set of equivalent martingale measures, by $\mathbb{T}$ the set of stopping times. Throughout this exercise, we want to prove that

$$\hat{P}_{\mathrm{a}}(0) = \sup_{(\tau,\mathbb{Q}) \in \mathbb{T} \times \mathcal{P}} \mathbb{E}^{\mathbb{Q}}[\xi(\tau)]. \tag{1}$$

(i) Show that $\geq$ holds in (1). *Hint:* The discounted wealth process of a self-financing trading strategy is a martingale.

(ii) Define the stochastic process U by

$$U(t) = \sup_{(\tau,\mathbb{Q}) \in \mathbb{T} \times \mathcal{P}} \mathbb{E}^{\mathbb{Q}}[\xi(\tau) \mid \mathfrak{F}(t)], \tag{2}$$

$t = 0, \dots, T$, and prove the following dynamic programming principle:

$$U(T) = \xi(T), \qquad \forall t = 1, \dots, T: \ U(t-1) = \xi(t-1) \vee \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}}[U(t) \mid \mathfrak{F}(t-1)]. \tag{3}$$

Hint: Compare with the dynamic programming principle discussed in the lecture. Note that in this exercise, we *define* the "envelope" U by equation (2) and *prove* (3), whereas at some similar point in the lecture we proceeded in the inverse order. But—as follows from the above assertions—this does not matter here.

(iii) Deduce from (3) that U is the smallest $\mathcal{P}$-supermartingale dominating[1] ξ.

(iv) Conclude that $\leq$ holds in (1).

Technische Universität Berlin, Fakultät II, Institut für Mathematik, Lehrstuhl für Stochastik und Quantitative Finanzmathematik, Strasse des 17. Juni 136, D-10623 Berlin

Email address: `profname@math.tu-berlin.de, assi1name@tu-berlin.de, assi2name@math.tu-berlin.de` [2]

[1] I.e. if Z is another $\mathcal{P}$-supermartingale with $\xi \leq Z$, then $U \leq Z$. A $\mathcal{P}$-supermartingale is a process which is a $\mathbb{Q}$-martingale for any $\mathbb{Q} \in \mathcal{P}$.

[2] The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".