HOMEWORK ASSIGNMENT NO. 10 — MATHEMATICAL FINANCE I

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Due date of this homework: 8 January 2020, 12 p.m. (noon), MA 141

The total score of this homework is 16 points (exercises one, two, three). This entire homework is not **compulsory**, but optional, so that you will obtain a correction and respective *bonus points* if you return your results until the above-mentioned time of delivery. We wish you Merry Christmas and a Happy New Year (cf. figure 1).

FIGURE 1. Santa Claus with Brownian beard as white noise (Licence of the picture: Creative Commons Attribution-Share Alike 3.0 Generic, source: `https://en.wikipedia.org/wiki/File:Sinterklaas_2007.jpg`, attribution: Gaby Kooiman, downloaded on 18 December 2019, at 12:30 p.m.)

Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. All or Nothing (6 bonus points)

Let be $r \in \mathbb{R}$, $\sigma > 0$, $s > 0$. In this exercise we consider the corresponding sequence $(\mathrm{CRR}(N))_{N\in\mathbb{N}}$ of CRR models as introduced in the lecture, namely, for any $N \in \mathbb{N}$ sufficiently large (see footnote 1 in the lecture notes, page 90), the CRR model $\mathrm{CRR}(N)$ is given by the parameters $r^N = \frac{r}{N}$, $u^N = \frac{r}{N} + \frac{\sigma}{\sqrt{N}}$, $d^N = \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$.

We want to derive continuous-time-limit expressions for the arbitrage-free prices of the following so-called (European) *binary options*—i.e. options with *discontinuous* payoff at maturity T—by computing the limit of the corresponding arbitrage-free prices obtained in the N-step CRR model as $N \longrightarrow \infty$.

Date: December 18, 2019.

Let $K > 0$ be a fixed strike price. For the three following European payoff schemes $(\xi_C^N)_{N\in\mathbb{N}}$, $(\xi_P)_{N\in\mathbb{N}}$, $(\xi_G)_{N\in\mathbb{N}}$ with maturity T, compute the sequences of their unique no-arbitrage prices at time 0 in the CRR(N) model $(C^N(0))_{N\in\mathbb{N}}$, $(P^N(0))_{N\in\mathbb{N}}$, $(G^N(0))_{N\in\mathbb{N}}$, and their limits, denoted by $C(0), P(0), G(0)$, for $N \longrightarrow \infty$, respectively:

(1) a *cash-or-nothing call option*—also called *digital call option*—on one unit of cash, written on S_N^1 with strike price K:

$$\xi_C^N = C^N(T) = \mathbf{1}_{S_N^1(T)>K};$$

(2) an *asset-or-nothing put option* with strike price K written on S_N^1:

$$\xi_P^N = P^N(T) = S_N^1(T)\mathbf{1}_{S_N^1(T)_N<K};$$

(3) a *gap call option* with strike price K and payoff strike $c > 0$, written on S_N^1:

$$\xi_G^N = G^N(T) = (S_N^1(T) - c)\mathbf{1}_{S_N^1(T)>K}.$$

2. Black-Scholes Formula and Implied Volatility (4 bonus points)

For any $r \in \mathbb{R}$, $s, \sigma, K, T > 0$, $s > 0$, we write

$$d_\pm(K,T,\sigma) = \frac{\log(s/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}},$$

and define $C(K,T,\sigma) = s\Phi(d_+(K,T,\sigma)) - K\,\mathrm{e}^{-rT}\,\Phi(d_-(K,T,\sigma))$, where Φ is the cumulative distribution function of the standard normal distribution.

Remark. Note that, if we consider the sequence $(\mathrm{CRR}(N))_{N\in\mathbb{N}}$ of CRR models as in the previous exercise, and if, for $N \in \mathbb{N}$ sufficiently large, we denote by $C^N(0)$ be the unique no-arbitrage price of a European call option on S_N^1 with maturity T an strike price K in the model CRR(N), then $C(K,T,\sigma) = \lim_{N\to\infty} C^N(0)$ (theorem 5.3, which *mutatis mutandis* holds also for $T \in \mathbb{R}_+^* \setminus \mathbb{N}$).

(1) Show that the "vega"[1] $\mathcal{V}$, i.e., the sensitivity of $C(K,T,\sigma)$ with respect to changes in the volatility σ, i.e.

$$\mathcal{V}(K,T,\sigma) = \frac{\partial}{\partial\sigma}C(K,T,\sigma),$$

is given by

$$\mathcal{V}(K,T\sigma) = s\sqrt{T}\varphi(d_+(K,T,\sigma))$$

where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}, x \mapsto \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$.

(2) Assume that, at time zero, a call option written on an asset S^1 with initial price $S^1(0) = s$, with strike K and maturity T, trades at some stock exchange at a price $C^*(K,T)$ which satisfies

$$(s - e^{-rT}K)_+ < C^*(K,T) < s.$$

Show that there exists a unique strictly positive real number, denoted by $\sigma^{\mathrm{imp}}(K,T)$, such that

$$C^*(K,T) = C(K,T,\sigma^{\mathrm{imp}}(K,T))$$

holds true. The value $\sigma^{\mathrm{imp}}(K,T)$ is called *implied volatility* of the call price $C^*(K,T)$ and is used exhaustively in practice since it readily allows to compare quoted call option prices with different strikes and maturities.

[1] Note that this is a fantasy name, but neither a greek nor another common letter, despite the fact that it is part of the family of "greeks", which are a frequently used tool in mathematical finance for describing the sensitivity of prices with respect to the model parameters.

3. Brownian Motion (6 bonus points)

Let W be a Wiener process on an adequate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ and $T > 0$.

(1) "Brownian Motion is Almost Never Monotone": Let $A \in \mathfrak{A}$ be the event of W being monotone increasing on some open sub-interval of $[0, T]$, i.e.

$$A = \Big\{\omega \in \Omega \mid \exists a, b \in [0, T]: \ (a < b, \forall s, t \in (a, b): \ (s \leq t \Rightarrow W(s, \omega) \leq W(t, \omega)))\Big\}.$$

Show that $\mathbb{P}(A) = 0$.

Hint: Show first that for any $a, b \in [0, T]$ with $a < b$, the event

$$B = \{\omega \in \Omega \mid \forall s, t \in (a, b): \ (s \leq t \Rightarrow W(s, \omega) \leq W(t, \omega))\}$$

has zero probability.

(2) Let be $\mu \in \mathbb{R}$, $\sigma > 0$. Write, for any $t \in [0, T]$, $X(t) = \mu t + \sigma W(t)$, $Y(t) = X(t)^2$, $Z(t) = \exp(X(t) - \frac{\sigma^2}{2} t)$. Give the necessary and sufficient conditions under which these continuous-time processes are martingales with respect to the augmented filtration $\mathfrak{F}^W$, respectively.

(3) Show that, for any $c > 0$, $t_0 \in (0, T)$, the following expressions in $t \in [0, T - t_0]$ also define Wiener processes:

$$-W(t), \qquad c^{-0.5} W(ct), \qquad W(t_0 + t) - W(t_0), \qquad W(T - t) - W(T).$$

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[2]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".