HOMEWORK ASSIGNMENT NO. 7 — MATHEMATICAL FINANCE I

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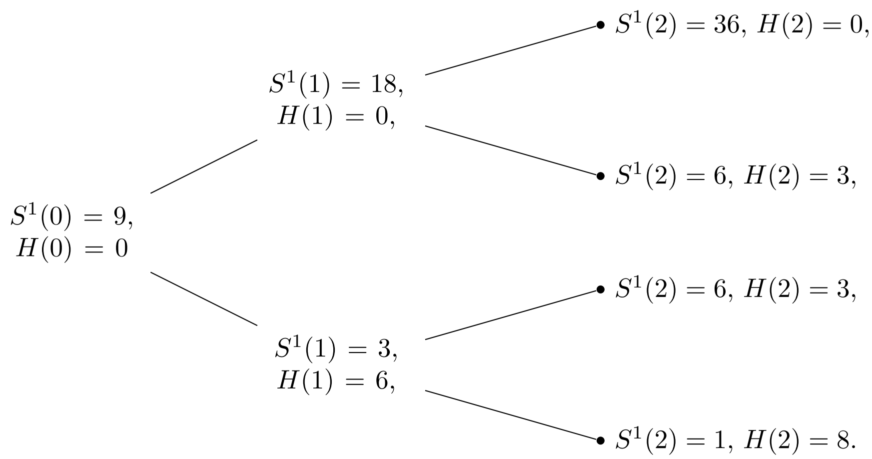
Return of this homework: 4 December 2019, 12 p.m. noon, MA 141

The total score of this homework is 16 points. The first and the second exercise give each 4 points, the third gives 8 points. In addition, there is an optional programming exercise which gives up to 8 *bonus points*.

Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Pricing and Replicating American Otions in the CRR Model (4 points)

In this exercise we consider a CRR model with the notations from the lecture notes with parameters $T = 2$ and $r = 0$ such that the movement of S^1 and of some American contingent claim denoted by H follow the following dynamics (up movements correspond to "u", down movements correspond to "d"):



(1) Calculate the model parameters u and d.
(2) Determine all equivalent martingale measures, and explain why there is at least one. You are also kindly asked to discuss its uniqueness.
(3) Further, solve the optimal stopping problem

$$V(0) = \sup_{\tau \in \mathbb{T}} \mathbb{E}^{\mathbb{Q}}[H(\tau)],$$

where $\mathbb{T}$ denotes the set of $\mathfrak{F}$-stopping times with values in $\mathcal{T}$.

Date: November 27, 2019.

2. To Choose or Not to Choose (4 points)

Let be $T \in \mathbb{N}$, $\mathcal{T} = \{0, \dots, T\}$, and $(\Omega, \mathfrak{A}, \mathbb{P})$ a probability space suitable for the following. Let X be a stochastic process on it with time $\mathcal{T}$. Let $\mathfrak{F}$ the "natural filtration" associated to the process X, i.e. the smallest filtration on the given probability space such that X is adapted to it. Let $\mathbb{T}$ be the set of $\mathfrak{F}$-stopping times with values in $\mathcal{T}$.

In the following, we consider the optimal stopping problem

$$V(0) = \sup_{\tau \in \mathbb{T}} \mathbb{E}[X(\tau)].$$

Suppose that $X(0), \dots, X(T)$ are i.i.d. copies of a uniformly distributed random variable on $(0, 1) \subset \mathbb{R}$. Then, solve the problem for fixed T, i.e. calculate $V(0)$ and provide an optimal stopping time. Discuss the dependence of the optimal stopping rule on T. What can you say about its asymptotic behaviour, i.e. its behaviour for $T \longrightarrow \infty$?

You are invited to give a – not necessarily economic (but possibly, if you prefer) – interpretation of this mathematical problem.

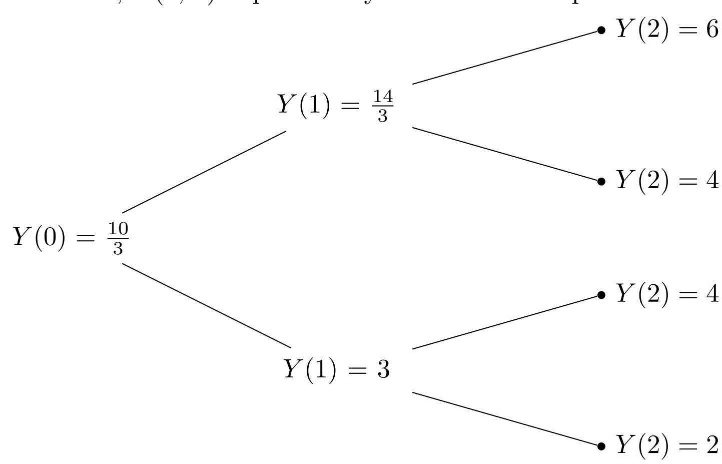
3. Optimal Stopping for an American Option (8 points)

Let us restate the *Doob decomposition* in discrete time on finite probability spaces. Let $\mathcal{T} = \{0, \dots, T\}$ for some $T \in \mathbb{N}$, $T > 0$, and $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ a finite filtered probability space with time $\mathcal{T}$. Then, every adapted stochastic process X on it admits a unique decomposition into a martingale M and a predictable process A with $A(0) = 0$, i.e.

$$X = M + A.$$

This has been proven in homework 3, exercise 2.

(1) Now we consider the case where $T = 2$, and $\Omega = \{u, d\}^2$, for some real numbers $u > d$. Consider a process Y given by the following tree (where up moves correspond to "u", down moves to "d"; $Y(1, \omega)$ depends only on the first component of $\omega \in \Omega$; $Y(0)$ is constant):



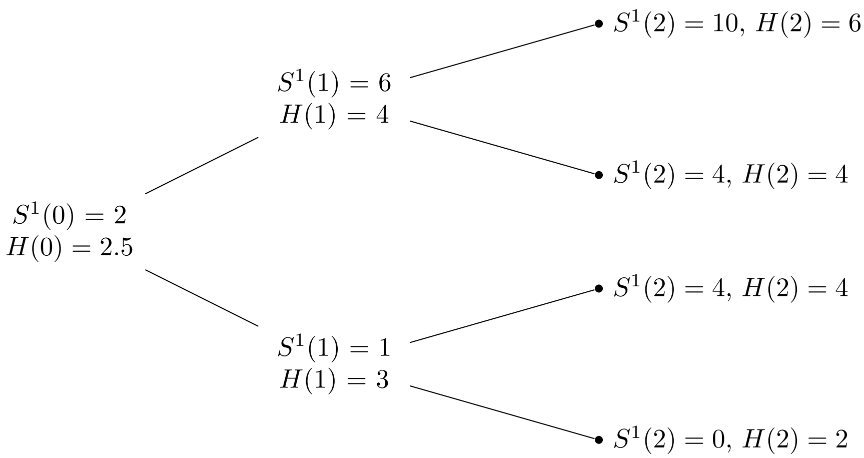
Let $\mathfrak{F} = \mathfrak{F}^Y$ the "natural filtration" associated to Y, let $\mathfrak{A} = \mathfrak{F}(2)$, and μ be the measure on $(\Omega, \mathfrak{A})$ defined through the formulae

$$\mu(\{(u,u)\}) = 1,$$
$$\mu(\{(u,d)\}) = 2,$$
$$\mu(\{(d,u)\}) = 3,$$
$$\mu(\{(d,d)\}) = 9,$$

and let $\mathbb{P}$ be its normalisation, i.e. $\mathbb{P} = \frac{1}{15}\mu$.

You are kindly asked to compute explicitly the Doob decomposition of Y.

(2) Now, we consider a finite financial market model with $T = 2$, $\Omega = \{u, d\}^2$ for two real numbers $u > d$, and given by a two-dimensional process $S = (S^0, S^1)$ and an American payoff process H, such that S^0 is constant and deterministic and S^1 and H are given via the following tree:



Let $\mathfrak{F} = \mathfrak{F}^S$ be the "natural filtration" associated to S, $\mathfrak{A} = \mathfrak{F}(2)$, and $\mathbb{P}$ such that $\mathbb{P}^{-1}(\{0\}) = \{\emptyset\}$.

(a) You are now kindly asked to compute the equivalent martingale measure $\mathbb{Q}$.

(b) Then, please specify explicitly the Snell envelope U of H with respect to $\mathbb{Q}$.

(c) Determine a superhedging strategy φ for H with initial capital $U(0)$, i.e. a (self-financing) trading strategy φ with initial wealth $U(0)$ such that

$$X^\varphi(t) \geq H(t),$$

for any $t = 0, 1, 2$.

(d) Calculate $\mathbb{E}^{\mathbb{Q}}[H(\tau)]$ for any $\tau \in \mathbb{T}$.

(e) Last, we assume that the American option has been sold at the price $\frac{10}{3}$ at time $t = 0$. Deduce from what has been showed above, at which nodes in the above tree executing the American option on the part of the holder guarantees a riskless profit for the seller.

4. Santa Claus Special: Pricing and Hedging American Options Numerically in the CRR Model (8 bonus points)

This exercise is **optional**; there is, of course, *no obligation to submit.* For the exercise assignment, see the extra sheet. You are kindly asked to hand in your solution to part 1 of the exercise together with the solutions to exercises 1 to 3.

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[1]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".