# HOMEWORK ASSIGNMENT NO. 6 — MATHEMATICAL FINANCE I

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*Return of this homework: 27 November 2019,* ***10 a.m.****, MA 141*

The total score of this homework is 16 points, each exercise giving rise to 4 points.
Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

**Please note**: The homework has to be returned at 10 a.m. on 27 November.

## 1. The density process

Let $(\Omega, \mathfrak{A}, \mathfrak{F})$ be a finite filtered measurable space with time $\mathcal{T} = \{0, \dots, T\}$, $T \in \mathbb{N}$, and $\mathbb{P}, \mathbb{Q}$ two equivalent probability measures on it. Let $Z$ be the $\mathfrak{F}$-$\mathbb{P}$-martingale defined by

$$Z(t) = \mathbb{E}^{\mathbb{P}}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}} \;\Big|\; \mathfrak{F}(t)\Big], \quad t \in \mathcal{T}.$$

You are kindly asked to prove the following statement:

A stochastic process $X$ on $\Omega$ and with time $\mathcal{T}$ is an $\mathfrak{F}$-$\mathbb{Q}$-supermartingale if—and only if—the product $XZ$ is an $\mathfrak{F}$-$\mathbb{P}$-supermartingale. The same holds after having replaced "supermartingale" by either "submartingale" or "martingale", respectively. (4 points)

## 2. Martingales and stopping times

Let $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ be a finite filtered probability space with time $\mathcal{T} = \{0, \dots, T\}$, $T \in \mathbb{N}$, and let $X$ be an adapted process on it. Please show the following four results!

(i) Let $\sigma$ and $\tau$ be stopping times with values in $\mathcal{T}$ and $\sigma \leq \tau$. If $X$ is a supermartingale, then

$$X(\sigma) \geq \mathbb{E}[X(\tau) \mid \mathfrak{F}(\sigma)]. \qquad \text{(1 point)}$$

(ii) $X$ is a supermartingale if—and only if—for any stopping times $\sigma, \tau$ with $\sigma \leq \tau$,

$$\mathbb{E}[X(\sigma)] \geq \mathbb{E}[X(\tau)]. \qquad \text{(1 point)}$$

(iii) Let $X = M + A$ be the Doob decomposition of $X$ into a martingale $M$ and a predictable process $A$ with $A(0) = 0$ (see homework assignment 3, exercise 2), and let $\tau$ be a stopping time with values in $\mathcal{T}$. Then, the Doob decomposition of the stopped process $X^\tau$ is given by

$$X^\tau = M^\tau + A^\tau. \qquad \text{(1 point)}$$

(iv) Let $X$ be a supermartingale and $\tau$ a stopping time. Then, $X^\tau$ is again a supermartingale, with respect to both $\mathfrak{F}$ and $\mathfrak{F}^\tau$, where, for each $t \in \mathbb{N}$, we put $\mathfrak{F}^\tau(t) = \mathfrak{F}(\tau \wedge t)$ (see homework assignment 3, exercise 4). (1 point)

Remark: The same statements hold—*mutatis mutandis* (sense of inequalities)—also for submartingales, and therefore, as well for martingales.

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*Date*: November 20, 2019.

## 3. Dynamic programming

We consider an arbitrage-free trinomial model with $T = 2$, using the notations from the lecture notes, and with $m = 0$. Please calculate the superhedging price of a European put option written on $S^1$, with strike $s = S^1(0)$ and maturity $T$ at time 0 via a backward induction using dynamic programming (see corollary 3.20)! (4 points)

## 4. Some optimal stopping

Let $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ be a finite filtered probability space with time $\mathcal{T} = \{0, \dots, T\}$, $T \in \mathbb{N}$, and let $H$ be an adapted process on it. Let $\mathbb{T}$ be the set of $\mathfrak{F}$-stopping times with values in $\mathcal{T}$.

(i) Please solve the optimal stopping problem

$$\sup_{\tau \in \mathbb{T}} \mathbb{E}[H(\tau)],$$

i.e. find $\tau^* \in \mathcal{A}$ which attains the supremum, in case of $H$ being

(a) a submartingale,

(b) a martingale,

respectively! (2 points)

(ii) Let $S = (S^0, S^1)$ be a finite arbitrage-free financial market on our space, with $d = 1$ and such that $S^0$ is monotone non-decreasing in time. We consider the payoff process $\xi$ of an American call option on $S^1$ with strike $K > 0$ and maturity $T$, i.e. $\xi = (S^1 - K)_+$. Find an optimal exercise strategy for the buyer, i.e. solve, for any equivalent martingale measure $\mathbb{Q}$, the optimal stopping problem

$$\sup_{\tau \in \mathbb{T}} \mathbb{E}^{\mathbb{Q}}[\bar{\xi}(\tau)]! \qquad \text{(2 points)}$$

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[1] The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".