HOMEWORK ASSIGNMENT NO. 5 — MATHEMATICAL FINANCE I

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Return of this homework: 20 November 2019, 12 noon, MA 141

The total score of this homework is 16 points, the first exercise giving rise to 8 points as well as the second one which is provided in an extra jupyter-file.

Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Some methods for calculating no-arbitrage prices

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ a finite probability space with time $\mathcal{T} = \{0, \ldots, T\}$, for some $T \in \mathbb{N}$, and with our usual conventions, i.e. $\mathbb{P}^{-1}\{0\} = \{\emptyset\}$, and $\mathfrak{A} = 2^\Omega$.

We let $S = (S^0, \ldots, S^d)$ be a complete finite financial market on this space together with its natural filtration $\mathfrak{F}^S$, and denote $\mathbb{Q}$ its equivalent martingale measure.

(i) Given a random vector Ξ alias an $\mathfrak{A}$-Borel-measurable map $\Omega \longrightarrow \mathbb{R}^n$ for some $n \in \mathbb{N}$, let ξ be a $\sigma(\Xi)$-measurable random variable. Please show, that there exists a measurable function $f_\xi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\xi = f_\xi \circ \Xi! \qquad \text{(1 point)}$$

(ii) Moreover, show that if Ξ' is another $\mathfrak{A}$-measurable random vector of dimension $n' \in \mathbb{N}$, then, for any Borel function $f : \mathbb{R}^{n+n'} \longrightarrow \mathbb{R}$, we have

$$\mathbb{E}\big[f(\Xi, \Xi') \mid \sigma(\Xi)\big](\omega) = \mathbb{E}\big[f(\Xi(\omega), \Xi') \mid \sigma(\Xi)\big](\omega),$$

for any $\omega \in \Omega$! (1 point)

(iii) Let ξ be an option. Let X be an $\mathfrak{F}^S$-adapted process such that $X(T) = \xi$. Recall why the extended market $(S^0, \ldots, S^d, X)$ is arbitrage-free if and only if

$$\overline{X}(s) = \mathbb{E}^{\mathbb{Q}}\big[\overline{X}(t) \mid \mathfrak{F}^S(s)\big],$$

for any $s, t = 0, \ldots, T$ with $s \le t$! (1 point)

(iv) We suppose that, with the notations from above, $d = 1$, and that the market follows a CRR model with T periods such that $\mathfrak{F}^S(T) = \mathfrak{A}$, $\mathfrak{F}^S(0) = \{\emptyset, \Omega\}$, and that we can suppose $\Omega = \{d, u\}^T$. Let $q \in (0, 1)$ be the number such that $\mathbb{Q}[R(1) = u] = q$. Given a European option $\xi \in L^0(\Omega, \mathfrak{F}^S(T))$, precedent homework and part (i) provide a measurable function $v_T : \mathbb{R}^T \longrightarrow \mathbb{R}$ such that $\xi = v_T(s, S^1(1), \ldots, S^1(T))$.

Please show that the recursive schemes called *backwards induction* and defined in (iva) and (ivb) allow for computing the time-zero no-arbitrage price π of ξ, for the first in general, for the second under the assumption that ξ is $\sigma(S^1(T))$-measurable, i.e. v_T admits a representation $w_T(x_T) = v_T(x_0, \ldots, x_T)$ for all $x_0, \ldots, x_T > 0$ for some Borel function w_T on $\mathbb{R}_+$!

(a) For $t = T, \ldots, 1$, define the function v_{t-1} by

$$v_{t-1}(x_0, \ldots, x_{t-1}) := q v_t(x_0, x_1, \ldots, x_{t-1}, x_{t-1}(1+u)) + (1-q) v_t(x_0, \ldots, x_{t-1}, x_{t-1}(1+d)),$$

for $x_0, \ldots, x_{t-1} > 0$, and set $\pi := v_0(s)$!

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(b) For $t = T, \dots, 1$, define the function w_{t-1} by

$$w_{t-1}(x_{t-1}) := qw_t(x_{t-1}(1+u)) + (1-q)w_t(x_{t-1}(1+d)),$$

for $x_{t-1} > 0$, and set $\pi := w_0(s)$!

(2 points)

(v) We define, for any $t = 1, \dots, T$, the function $\Delta_t : (0,\infty)^{t+1} \longrightarrow \mathbb{R}$ by

$$\Delta_t(x_{t-1}) = (1+r)^t \frac{v_t(x_0, \dots, x_{t-1}, x_{t-1}(1+u)) - v_t(x_0, \dots, x_{t-1}, x_{t-1}(1+d))}{x_{t-1}(u-d)},$$

$x_1, \dots, x_{t-1} > 0$. Then show that the process φ defined by

$$\varphi(t,\omega) = \Delta_t(s, S^1(1,\omega), \dots, S^1(t-1,\omega)),$$

for $t = 1, \dots, T$, $\omega \in \Omega$, defines the unique replication strategy for ξ! Due to its difference quotient's nature, the function Δ_t is called the *delta* of S^1 at time t and φ the corresponding *delta hedge*. (1 point)

(vi) Now, we consider the case where ξ is even $\sigma(S^1(T))$-measurable. With the aid of a combinatorial argument, show that

$$\mathbb{E}^{\mathbb{Q}}[\bar{\xi}] = \sum_{t=0}^{T} \binom{T}{t} q^t (1-q)^{T-t} \bar{\xi}(\omega_t),$$

where, for each $t = 0, \dots, T$, ω_t denotes an arbitrary element of Ω satisfying $S^1(T,\omega_t) = s(1+u)^t(1+d)^{T-t}$! This yields another method for computing the no-arbitrage price. (1 point)

(vii) Let $X_1, \dots, X_k$ be $k \in \mathbb{N}$ i.i.d. copies of $\bar{\xi}$. Then, the sequence $(S_n)_n$, where $S_n = \frac{X_1 + \cdots + X_n}{n}$ for $n \in \mathbb{N} \setminus \{0\}$, converges pointwise to $\mathbb{E}^{\mathbb{Q}}[\bar{\xi}]$. From which fundamental mathematical result does this follow? This gives us another approach to calculating the no-arbitrage price of ξ (called: the *Monte Carlo* method). (1 point)

2. Programming

For the task of this exercise, please consult the uploaded jupyter-file!

For help with jupyter notebook, you may type "real python jupyter notebook introduction" on a search engine of your personal preference.

For help with python, you may consult the online python documentation ("python docs").

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[1] The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".