HOMEWORK ASSIGNMENT NO. 4 — MATHEMATICAL FINANCE I

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Return of this homework: 13 November 2019, 12 noon, MA 141

The total score of this homework is 16 points, each exercise giving rise to 4 points.
Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Attainability

(i) With the notations from the lecture notes, we consider the CRR model and an option $\xi = g(S^1(T))$ for some function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$.
  (a) Please construct a replication strategy for ξ in case of $T = 1$. (1 point)
  (b) Please construct a replication strategy for ξ in case of $T = 2$ and $g(x) = (x - K)_+$ for any $x \in \mathbb{R}$ (European call on S^1 with strike K and maturity T), where $K = s(1 + d)(1 + u)$. (1 point)

(ii) With the notations from the lecture notes, we consider the trinomial model with $T = 1$ and fix $K > 0$ with $s(1 + d) < K < s(1 + u)$ in order to examine the payoffs $\xi_P = (K - S^1(1))_+$ and $\xi_F = S^1(1) - K$. Please argue whether these payoffs are attainable or not! (2 points)

2. Replication and superreplication

(i) Let be given a finite financial market S with time $\{0, 1, \dots, T\}$ for some $T \in \mathbb{N}$, defined on a finite probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ as in definition 2.7 of the lecture notes, such that all assets are deterministic and constant numéraires (in the sense of definition 2.4). Please calculate the superhedging price of an arbitrary option ξ! (1 point)

(ii) We consider a one-period financial market S on a finite probability space, and with two assets (especially, we have $d = 1$ in definition 2.7). Please express the superhedging price of a European call option on S^1 in terms of its strike, $\sup S^1(1)$ and $\inf S^1(1)$! (1 point)

(iii) We consider a market $S = (S^0, S^1)$ with $\Omega = \{\omega_1, \omega_2\}$, $T = 1$ and $S^0(0) = S^0(1) = 1$. We assume that trading in the security S^1 incurs transaction cost, which is to say that S^1 is bought at the ask (or offer) price S^{a} and sold at the bid price S^{b} with

$$\begin{aligned} &S^{\mathrm{a}}(0) = 5, \quad S^{\mathrm{a}}(1, \omega_1) = 3, \quad S^{\mathrm{a}}(1, \omega_2) = 6, \\ &S^{\mathrm{b}}(0) = 3, \quad S^{\mathrm{b}}(1, \omega_1) = 2, \quad S^{\mathrm{b}}(1, \omega_2) = 4. \end{aligned}$$

We assume moreover that any position in S^1 has to be liquidated at time T (or, put another way, the wealth of a share in S^1 has to be computed in terms of S^{b} or S^{a} for a long or a short position, respectively). Given the option ξ defined by $\xi(\omega_1) = 0$ and $\xi(\omega_2) = 2$, please construct both
  (a) a replication strategy for ξ, (1 point)
  (b) and a superhedging strategy for ξ requiring strictly less initial capital than this replication strategy! (1 point)

Date: November 6, 2019.

3. Completeness and the trinomial model

In this exercise, we consider the one-period trinomial model as defined in the lecture notes (i.e. with $T = 1$) and write $R = R_1$. For simplicity, we assume that $m = 0$. Then, we suppose that the market is arbitrage-free which—as has been showed in the last homework—is equivalent to $r \in (d, u)$.

(i) Let be $q_u, q_m, q_d \in (0, 1)$ with $q_u + q_m + q_d = 1$, and define a probability measure $\mathbb{Q}$ by $\mathbb{Q}[R = x] = q_x$ for $x \in \{u, m, d\}$.
  (a) Please show that $\mathbb{Q}$ is an equivalent martingale measure if and only if $\mathbb{E}^{\mathbb{Q}}[R] = r$!
  (b) Then, deduce from this that there exists a maximal $q^* \in (0, 1]$ such that for all $q \in (0, q^*)$, the measure $\mathbb{Q}$ defined by some $(q_u, q_m, q_d) \in (0, 1)^3$ such that $q_m = q$ is an equivalent martingale measure if and only if

$$q_u = \frac{r - (1 - q_m)d}{u - d}, \qquad q_d = \frac{u(1 - q_m) - r}{u - d}, \tag{1}$$

  and calculate q^*!
  (c) Conclude that the market is not complete!
  (1.5 points)

(ii) For $q \in (0, q^*)$, we denote by $\mathbb{Q}_q$ the measure defined by the condition (1). Denote ξ the payoff given by a put option on S^1 with strike $K \in (s(1+d), s)$, i.e. $\xi = (K - S^1(1))_+$. Please compute and interpret the quantities

$$\check{\pi} = \inf_{q \in (0,q^*)} \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}_q}[\xi], \qquad \hat{\pi} = \sup_{q \in (0,q^*)} \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}_q}[\xi]$$

and, in particular, establish the inequality $\check{\pi} < \hat{\pi}$ (1 point)!

(iii) Let us extend the aforementioned market by the option ξ, for some arbitrary $K \in (s(1+d), s)$ and with initial price $\pi \in (\check{\pi}, \hat{\pi})$, i.e. we consider the market $S = (S^0, S^1, S^2)$, where $S^2(0) = \pi$ and $S^2(1) = \xi$.
  (a) Please show that, in this market, any option is attainable,
  (b) determine the unique equivalent martingale measure for S by calculating the respective $q \in (0, 1)$ in terms of s, u, d, r, π,
  (c) and calculate the risk-neutral price of a call option on S^2 with strike $L \in (0, K - s(1 + d))$!
  (1.5 points)

4. Filtrations and insider trading

For $T \in \mathbb{N}$, we consider a T-period market model on a finite probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with $\mathbb{P}^{-1}(\{0\}) = \{\emptyset\}$ with two assets (S^0, S^1) where S^0 is constantly equal to one and

$$S^1(t) = 1 + \sum_{s=1}^{t} Y(s)$$

for $s = 0, 1, \ldots, T$, where $Y(1), \ldots, Y(T)$ are i.i.d. copies of some random variable $\hat{Y}$ on $(\Omega, \mathfrak{A}, \mathbb{P})$ with $\mathbb{P}[\hat{Y} > 0] > 0$, $\mathbb{E}[\hat{Y}] = 0$ and $\hat{Y} \geq -1/T$.

We consider the enlarged filtration $\mathfrak{F}$ given by

$$\mathfrak{F}(t) = \sigma(\mathfrak{F}^S(t), S^1(T))$$

for $t = 0, 1, \ldots, T$.

(i) Please show that S^1 is a martingale with respect to $\mathfrak{F}^S$, but not with respect to $\mathfrak{F}$! (1 point)

(ii) Show that the stochastic process X defined by

$$X(t) = S^1(t) - \sum_{s=0}^{t-1} \frac{S^1(T) - S^1(s)}{T - s},$$

$t = 0, \dots, T$, is a martingale with respect to $\mathfrak{F}$! (1 point)

(iii) Show that the market $(S, \Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ admits an arbitrage possibility φ! Nota bene: here, we consider the filtration $\mathfrak{F}$, not $\mathfrak{F}^S$! (1 point)

(iv) Please give an economic interpretation of the filtration $\mathfrak{F}$ as well as of φ! (1 point)

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[1]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".