HOMEWORK ASSIGNMENT NO. 3 — MATHEMATICAL FINANCE I

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Return of this homework: 6 November 2019, 12 noon, MA 141

The total score of this homework is 16 points, each exercise's total score being 4 points. Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Arbitrage and risk-neutral pricing

(i) Please find a condition on the parameters d, m, u, r in the trinomial model which is equivalent to the statement that the model is arbitrage-free and prove it! (1 point)

(ii) Suppose the parameters are chosen such that the trinomial model is arbitrage free. Please construct an equivalent martingale measure, and, given $T = 1$ and a self-financing trading strategy φ, provide an expression for $X^{\varphi}(0)$ in terms of the parameters d, m, u, r and $\varphi(T)$. (2 points)

(iii) With the notations from the lecture notes, give an example of an arbitrage-free finite financial market S with time horizon $T \in \mathbb{N}$ and an option ξ alias a non-negative random variable such that the price

$$P(t) = S_0(t)\mathbb{E}_t^{\mathbb{P}}\Big[\frac{\xi}{S^0(T)}\Big],$$

for $t = 0, \dots, T$, leads to arbitrage opportunities in the extended financial market (S, P), i.e. supposing that the option ξ can be traded under the price $P(t)$ at time t. Here $\mathbb{P}$ denotes the so-called *physical* or *objective measure* of the financial market, i.e. the measure which is contained right in its definition. Please use a simple example by choosing the CRR model with $r = 0$ and $T = 1$! (1 point)

2. Stochastic processes

Let $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ be some finite filtered probability space with time $\mathbb{N}$ and such that $\mathbb{P}^{-1}(\{0\}) = \{\emptyset\}$. Let $X : \mathbb{N} \times \Omega \longrightarrow \mathbb{R}$ be an adapted stochastic process on it.

(i) Please show that X can be uniquely decomposed into a sum

$$X = M + A + \xi,$$

where ξ is an $\mathfrak{F}(0)$-measurable random variable, and M is a martingale and A a predictable process satisfying $M(0) = A(0) = 0$! (3 points)

(ii) Then, deduce that X is a submartingale if and only if the process A of the above decomposition is monotone increasing! (1 point)

Date: October 30, 2019.

3. Measure theory

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be some probability space and $\mathfrak{B} \subset \mathfrak{A}$ and $\mathfrak{C} \subset \mathfrak{B}$ some sub-σ-algebras.

(i) Let be $\xi \in L^2(\Omega, \mathfrak{A}, \mathbb{P})$. Show that $\mathbb{E}[\xi \mid \mathfrak{B}]$ is the orthogonal projection of X onto the subspace $L^2(\Omega, \mathfrak{B}, \mathbb{P})$ with respect to the L^2-scalar product[1]! (1 point)

(ii) In this sub-exercise, we suppose Ω to be finite, and denote $\mathcal{P}$ the generating partition of $\mathfrak{B}$. Then, please show, that

$$\mathbb{E}[\xi \mid \mathfrak{B}] = \sum_{P \in \mathcal{P}} \mathbf{1}_P \cdot \mathbb{E}[\xi \mid P]. \qquad \text{(1 point)}$$

Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathfrak{A})$ which is equivalent[2] to $\mathbb{P}$.

(a) By the Radon-Nikodým theorem, $\mathbb{P}$ has an $\mathbb{Q}$-integrable density with respect to $\mathbb{Q}$ (and vice versa). Please recall why it is $\mathbb{Q}$-a.s. unique, and show that, in our special situation, it is $\mathbb{Q}$-a.s. strictly positive! (1 point)

(b) Let μ, ν, ρ be three probability measures on $(\Omega, \mathfrak{A})$ such that $\mu \ll \nu \ll \rho$. Denote by f and g the densities of μ with respect to ν and ν with respect to ρ, respectively. Prove that the density of μ with respect to ρ is (a.s.) given by $f \cdot g$! Please deduce that the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is (a.s.) given by $1/f$! (1 point)

4. The optional sampling theorem

Write $\mathcal{T} = \{0, \ldots, T\}$ for some $T \in \mathbb{N}$. Let $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ be a filtered probability space with time $\mathcal{T}$, $X : \mathcal{T} \times \Omega \longrightarrow \mathbb{R}$ a martingale and $H : \mathcal{T} \times \Omega \longrightarrow \mathbb{R}$ a predictable process on it. The stochastic integral $H \bullet X$ is defined to be the stochastic process given by

$$(H \bullet X)(t) = \sum_{s=1}^{t} H(s)(X(s) - X(s-1)), \quad t \in \mathcal{T}.$$

(i) Please show that, if H is bounded, then $H \bullet X$ is a martingale! (1 point)

(ii) Let $\sigma, \tau : \Omega \longrightarrow \mathcal{T}$ be stopping times (on the defined probability space), i.e. random variables ρ taking values in the set of considered times $\mathcal{T}$ such that $\{\rho \leq t\} \in \mathfrak{F}(t)$ for any $t \in \mathcal{T}$, and such that $\sigma \leq \tau$. We write $\mathfrak{F}(\sigma)$ the induced σ-algebra given by

$$\mathfrak{F}(\sigma) = \{A \in \mathfrak{A} \mid \forall t \in \mathbb{N} : A \cap \{\sigma \leq t\} \in \mathfrak{F}(t)\}.$$

Please show that it is a σ-algebra, which, for any time $s \in \mathcal{T}$, coincides with $\mathfrak{F}(s)$ (in the usual, standard sense) if $s = \sigma$! Moreover, please argue why $X(\rho)$, defined by $X(\rho)(\omega) = X(\rho(\omega), \omega)$ for $\omega \in \Omega$, is a random variable for any stopping time ρ! (1 point)

(iii) Then, and with the notations from above, please construct, for any $F \in \mathfrak{F}(\sigma)$, a bounded predictable process $H_F : \mathcal{T} \times \Omega \longrightarrow \mathbb{R}$ such that

$$H_F \bullet X = \mathbf{1}_F \cdot (X^\tau - X^\sigma),$$

where, for any stopping time ρ on our probability space, we define the *stopped process* X^ρ by $X^\rho(t) = X(\rho \wedge t)$, for any $t \in \mathbb{N}$. Here, $\wedge$ denotes the minimum operator! (1 point)

(iv) Deduce that

$$X(\sigma) = \mathbb{E}\big[X(\tau) \mid \mathfrak{F}(\sigma)\big]! \qquad \text{(1 point)}$$

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[1] Given a Hilbert space V, then for any $v \in V$ and any closed linear subspace $W \subset V$, there exists a unique $\bar{v} \in W$ such that $\langle w, v - \bar{v} \rangle = 0$ for all $w \in W$. This $\bar{v}$ is called the orthogonal projection of v onto W.

[2] ... with respect to the equivalence relation induced by the partial order $\ll$ of absolute continuity.

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[3]The strings preceding "@" have to be replaced by the corresponding family names. Please replace the symbol eszett by "ss".