HOMEWORK ASSIGNMENT NO. 2 — MATHEMATICAL FINANCE I

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Return of this homework: 30 October 2019, 12 noon, MA 141

The total score of this homework is 16 points, each exercise's total score being 4 points.
Please note that in all the homework assignments of this course, we assume that $0 \in \mathbb{N}$.

1. Adapted and predictable processes

We consider a finite financial market model with two assets S^0 and S^1 defined on a filtered probability space $(\Omega, \mathfrak{A}, (\mathfrak{F}(t))_{t\in\mathcal{T}}, \mathbb{P})$, where Ω can be assumed to be finite and $\mathcal{T} = \{0, 1, \dots, T-1, T\}$ for some strictly positive $T \in \mathbb{N}$. Please argue whether the trading strategies given by the following formulae, to be read for all $t \in \mathcal{T} \setminus \{0\}$ respectively, are either adapted, predictable, or none of them!

(i) $\varphi_1(t) = \mathbf{1}_{S^0(t) > S^1(t-1)}$ (1 point)

(ii) $\psi_1(t) = \begin{cases} 1, & \text{if } t = 1, \\ \mathbf{1}_{S^1(t-1) < S^1(t-2)}, & \text{else,} \end{cases}$ (1 point)

(iii) $\chi_1(t) = \begin{cases} 0, & \text{if } t \le t_0, \\ \mathbf{1}_A, & \text{else,} \end{cases}$, where $(t_0, A) \in \mathcal{T} \times \mathfrak{F}(t_0)$ (1 point)

(iv) $\xi_1(t) = \left(\sup_{s\in\mathcal{T}} S^1(s) - S^1(t)\right)_+$ (1 point)

2. Filtrations

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a finite probability space with $\mathfrak{A}$ being the power set of Ω, and $(\mathfrak{F}(t))_{t\in\mathbb{N}}$ a family of sub-σ-algebras of $\mathfrak{A}$.

(i) Denote by $\mathcal{P}(t)$ the partition associated to the sub-σ-algebra $\mathfrak{F}(t)$ for each $t \in \mathbb{N}$ (cf. exercise 1, homework 1). Please argue why $(\mathfrak{F}(t))_{t\in\mathbb{N}}$ is a filtration if and only if for all $t, s \in \mathbb{N}$ with $t < s$ the partition $\mathcal{P}(s)$ is a refinement of $\mathcal{P}(t)$! (1 point)

(ii) We consider a binomial model on $(\Omega, \mathfrak{A}, \mathbb{P})$ with $T \in \mathbb{N} + 1$ periods and assets $S = (S^0, S^1)$, and denote the corresponding Bernoulli random variables $R(1), \dots, R(T) > -1$, i.e.

$$S^1(t) = S^1(t-1)(1 + R(t))$$

for $t = 1, \dots, T$, and $S^1(0) = s$ for some $s > 0$, and $S^0(t) = (1+r)^t$ for all $t = 0, \dots, T$. Please show that the natural filtration $(\mathfrak{F}^S(t))_{t=0,\dots,T}$ is given by

$$\mathfrak{F}^S(t) = \sigma(R(1), \dots, R(t))$$

for all $t = 0, \dots, T$! (1 point)

(iii) In the aforementioned binomial model, please interpret the conditions $\mathfrak{F}^S(0) = \{\emptyset, \Omega\}$ and $\mathfrak{F}^S(T) = \mathfrak{A}$ by arguing which real-valued functions on Ω are $\mathfrak{F}^S(t)$-measurable for $t \in \{0, T\}$, respectively, and by deducing the cardinality of Ω from the second one! (1 point)

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(iv) In the same model, we put $T = 3$. We let $u > d > -1$ be the returns of up and down move as defined in the lecture, and suppose that $\Omega = \{d, u\}^3$ and

$$R(t)(\omega) = \omega_t$$

for any $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega$, $t = 1, 2, 3$. Please calculate explicitly both the natural filtration and the corresponding family of partitions! Visualise the filtration! (1 point)

3. Self-financing trading strategies

We consider a financial market with time $\mathcal{T} = \{0, 1, \ldots, T-1, T\}$, $T \in \mathbb{N}+1$, and with one bond S^0 and one further asset S^1 on a finite probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ such that $\mathbb{P}^{-1}(\{0\}) = \{\emptyset\}$ and equipped with the natural filtration. We suppose that the price of S^0 is constant and equal to one, that $S^1 > 0$ and $S^1(0) = s$ for some $s > 0$. Let $\mathcal{P}$ be the set of all trading strategies φ whose wealth process X^φ is strictly positive. Let F be the map from $\mathcal{P}$ to the set of two-dimensional predictable processes of our financial market defined by

$$(\varphi_0, \varphi_1) \mapsto \left(\frac{\varphi_0(t)}{X^\varphi(t-1)}, \frac{\varphi_1(t) S^1(t-1)}{X^\varphi(t-1)} \right)_{t=1,\ldots,T}.$$

(i) Given $\varphi \in \mathcal{P}$, please give an economic interpretation of $F(\varphi)$! (1 point)

(ii) We fix $x > 0$ and let $\mathcal{S}_x \subset \mathcal{P}$ be the set of self-financing trading strategies contained in $\mathcal{P}$ with initial wealth x. Please show that for any $\varphi \in \mathcal{S}_x$ its image $F(\varphi)$ alias $\pi = (\pi_0(t), \pi_1(t))_{t=1,\ldots,T}$ satisfies

$$\forall t = 1, \ldots, T: \quad \pi_0(t) + \pi_1(t) = 1, \quad R^\varphi(t) = \pi_1(t) \cdot R^1(t), \qquad \pi_1(1) = s \cdot \frac{\varphi_1(1)}{x}, \tag{1}$$

where we denote R^φ and R^1 the returns of the portfolio defined by φ and S^1, respectively, that is:

$$R^\varphi(t) = \frac{X^\varphi(t) - X^\varphi(t-1)}{X^\varphi(t-1)}, \qquad R^1(t) = \frac{S^1(t) - S^1(t-1)}{S^1(t-1)},$$

$t = 1, \ldots, T$!

(iii) Please prove that F induces a bijection between $\mathcal{S}_x$ and the set of the two-dimensional predictable processes $\pi = (\pi_0(t), \pi_1(t))_{t=1,\ldots,T}$ that satisfy (1) for some $\varphi \in \mathcal{S}_x$! (1 point)

(iv) Please express the wealth process of any $\varphi \in \mathcal{S}_x$ in terms of (x, π_1, R^1), where $\pi = (\pi_0, \pi_1)$ is the process corresponding to φ via the map F! (1 point)

4. Replication

(i) We consider a binomial financial market model with only two time points, 0 and 1, and two assets, one zero-coupon bond S^0 and one stock S^1. We denote $\mathbb{P}$ the probability measure of our model. The prices at 0 are supposed to be 1 and 10, respectively, and we know in advance that

$$\mathbb{P}[S^1(1) = 12] = 0.55, \qquad \mathbb{P}[S^1(1) = 8] = 0.45.$$

Please construct a self-financing portfolio which *replicates*[1] a put option P on S^1 with maturity 1 and strike 10, calculate the expected wealth at time 1 of a trader buying one put P and selling $P(0)$ bonds at time 0 for the following four scenarios

(a) $P(0) = 0.8$,
(b) $P(0) = 0.9$,
(c) $P(0) = 1.0$,
(d) $P(0) = 1.1$,

[1]... that is its wealth coincides with the value of P at time 1.

and explain in which of these scenarios there is an arbitrage opportunity in the extended market (S^0, S^1, P)! When would an arbitrageur alias a rational agent who wants to make riskless profits, invest? (3 points)

(ii) We consider the trinomial model with two time points and one bond S^0 and one stock S^1. For the deterministic bond we have

$$S^0(0) = 1, \qquad S^0(1) = 1 + r,$$

where $r > -1$, and for the stock we assume that

$$S^1(0) = 1, \qquad S^1(1) = \begin{cases} 1+u, & \text{with probability } p_u, \\ 1+m, & \text{with probability } p_m, \\ 1+d, & \text{with probability } p_d, \end{cases}$$

where $u > m > d > -1$, and $p_u, p_m, p_d > 0$ such that $p_u + p_m + p_d = 1$. Please show that there exists some *contingent claim* which is not *attainable*[2]! (1 point)

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[2]... that is: there exists a payoff function alias a vector $(x_u, x_m, x_d) \in \mathbb{R}^3$ which cannot be replicated by any self-financing portfolio of our market, i.e. there is no self-financing market portfolio such that its value at time 1 coincides with x_y if $S(1) = 1 + y$, for all $y \in \{u, m, d\}$.

[3]The strings preceding “@” have to be replaced by the corresponding family names. Please replace the symbol eszett by “ss”.