HOMEWORK ASSIGNMENT NO. 1 — MATHEMATICAL FINANCE I

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Return of this homework: 23 October 2019, 12 noon, MA 141

The total score of this homework is 16 points, each exercise giving rise to exactly 4 points.

1. Finite measurable spaces

Let $(\Omega, \mathfrak{A})$ be a measurable space whose sample space Ω has finite cardinality.

(i) Please show that $\mathfrak{A}$ is generated by a unique partition $\mathcal{P}$ of Ω! (1 point)

(ii) Let $\mathfrak{B}, \mathfrak{B}'$ be sub-σ-algebras of $\mathfrak{A}$ and $\mathcal{P}, \mathcal{P}'$ be the corresponding partitions (see the first question). Please show that $\mathfrak{B} \subset \mathfrak{B}'$ if and only if $\mathcal{P}'$ is a refinement of $\mathcal{P}$ in the sense that $\mathcal{P}'$ is the union of some set containing exactly one partition of any element of $\mathcal{P}$! (1 point)

(iii) Please deduce that a function $f : \Omega \to \mathbb{R}$ is Borel-measurable with respect to $\mathfrak{A}$ if and only if it is constant on each member of the generating partition $\mathcal{P}$! Put differently, you may show that the set Borel functions on $(\Omega, \mathfrak{A})$ is equal to the span of $\{\mathbf{1}_A \mid A \in \mathcal{P}\}$ over $\mathbb{R}$! (1 point)

(iv) We denote $2^{\mathcal{P}}$ the power set of $\mathcal{P}$. Please explain why there exists a bijection from the set of probability measures on $(\Omega, \mathfrak{A})$ to the set of probability measures on $(\mathcal{P}, 2^{\mathcal{P}})$ such that – in the case that $|\mathcal{P}| = |\Omega|$ – it reduces to the map induced by the identity mapping! (1 point)

Remark: We conclude that when discussing finite measurable spaces it suffices – in the sense we specified – to consider only those where the σ-algebra is the biggest one possible, i.e. given by the power set.

2. Financial engineering

In this exercise we consider a financial market from a point of view which is as model-free as possible, in the sense of a platform where some stock can be bought and sold at one unique price at each time instant in arbitrary quantity, idem for any option written on it. Time may be imagined as some subset of the real interval $[0, T]$ containing the starting point 0 and the maturity $T > 0$, prices as positive real numbers.

By building a portfolio of suitable European call and put options please construct and draw the payoff diagram of an option with a final payoff that, respectively,

(i) leads to a gain if the underlying stock price at maturity differs by a large amount from today's stock price, (1 point)

(ii) leads to an increasing gain if the stock price at maturity is slightly higher than today's price and to a constant gain if it is much higher than today's price! (1 point)

Please deduce what expectations with regard to the stock price two speculators – who prefer more to less – following the subsequent strategies have and visualise these via payoff diagrams:

(i) *bearish spread*: buy one European call and sell a second one with the same expiry date, but a smaller strike price, (1 point)

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(ii) *strangle*: buy one European put and buy one European call with the same expiry date, but a larger strike price! (1 point)

3. The no-arbitrage assumption

In this exercise, we always consider a financial market with no arbitrage opportunities.

(i) We now fix a maturity $T > 0$ and suppose that $B(t, T) \leq 1$ for all future times $t \leq T$. Given $T_1 < T_2 \leq T$ two future times, denote by C_1 and C_2 European calls on the same underlying with the same strike price but differing maturities, T_1 and T_2, respectively. Please show that $C_1(t) \leq C_2(t)$ for all $t \leq T_1$! (1 point)

(ii) A *digital call* (resp. *digital put*) with maturity T and strike $K > 0$ on an underlying S pays a fixed amount of money – say $A > 0$ – at time T if $S(T) \geq K$ (resp. $S(T) < K$). Here, we denote C_{dig} (resp. P_{dig}) the price process of such a digital call (resp. put) option. Please state and prove a put-call parity for digital options, that is find $x \in \mathbb{R}^4$ with $x_1, x_2 \neq 0$ such that for any underlying S and any zero-coupon bond B with maturity T holds

$$\forall t \in \mathcal{T}: \; x_1 C_{\text{dig}}(t) + x_2 P_{\text{dig}}(t) + x_3 S(t) + x_4 B(t, T) = 0,$$

where $\mathcal{T} \subset [0, T]$ denotes the (finite) set of considered points in time containing at least 0 and T. (1 point)

(iii) Let again $\mathcal{T} \subset [0, T]$ denote the (finite) set of considered points in time containing at least 0 and the maturity $T > 0$. We suppose here that our market contains a zero-coupon bond B with non-negative interest rates such that the function $\mathcal{T} \to \mathbb{R}, t \mapsto B(t, T)$ is monotone increasing. Then, given an American call C^a and an American put P^a, each with maturity T, some strike $K > 0$, and written on a stock S, please prove that

$$C^a(t) - P^a(t) \geq S(t) - K$$

for all times $t \in \mathcal{T}$! (1 point)

(iv) Given the following price quotes in today's newspaper (t = today, T = tomorrow) for some cash amount $K > 0$:

$$S(t) = 67.13, \qquad \text{a share of some stock company,}$$
$$C(t) = 11.42, \qquad \text{a European call on } S\text{, maturity } T\text{, strike } K,$$
$$P(t) = 3.94, \qquad \text{a European put on } S\text{, maturity } T\text{, strike } K,$$
$$B(t, T) = \exp(-0.02), \qquad \text{a zero-coupon bond, maturity } T,$$
$$P^a(t) = 5.67, \qquad \text{an American put on } S\text{, maturity } T\text{, strike } K,$$

is there an arbitrage opportunity? Please give the conditions under which this is the case, if there are, and provide the trading strategy required to exploit it! (1 point)

4. Arbitrage, speculation, and hedging

(i) Given a stock which is dually listed both on Frankfurter Wertpapierbörse (FWB) and on London Stock Exchange (LSE). Suppose that anywhere one euro is traded at 0.90 pound sterling, and that the stock price in pound sterling at LSE is 0.9 times its price in euro at FWB. Now assume, that an instant later the pound sterling suddenly appreciates versus the euro such that one euro is henceforth traded at 0.85 pound sterling, the stock prices having however stayed at the same level. Is there an arbitrage opportunity? If yes, which one, and what is going to happen to the stock prices as a result of potential arbitrage trades according to the law of supply and demand? (1 point)

(ii) Suppose the current price of a stock to be 50 euro and that the European call options with a maturity of two months and with a strike of 51 euro currently sell for 2.50 euro. Given an investor speculating on a rising stock price wants to invest $10,000$ euro, please compare the two strategies to buy 200 shares or $4,000$ options! How high does the stock price have to rise for the option strategy to be more profitable? What are the net losses, if the stock prices stay at 50 or fall to 45 euro for either of the two strategies? (2 points)

(iii) An investor owns $1,000$ shares of a company and wants to limit downside risk. Their current price is 100 euro and the European put option with a maturity of 6 months and a strike of 90 euro costs 6.50 euro. He thinks about choosing one of the following strategies: Either buying $1,000$ of the above put options or placing a so-called “stop loss order” at 90 euro i.e. instructing a broker to sell all the shares automatically once their price reaches 90 euro. Discuss the gain and risk profiles of both strategies. (1 point)

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[1]The strings preceding “@” have to be replaced by the corresponding family names. Please replace the symbol eszett by “ss”.