EXERCISE NO. 1 — MATHEMATICAL FINANCE I

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Material for the exercise classes on 23 and 25 October 2019
(this is not a homework assignment, a fact that has not necessarily to dishearten you from having a look at it)

1. Revising Euclidean spaces – the hyperplane separation theorem

Please show the following results for any Euclidean space V of finite, strictly positive dimension by only using the finite-dimensional Euclidean spaces theory[1]!

(i) Given two disjoint convex subsets C, D, there exist a $v \in V \setminus \{0\}$ and a real number $\beta \in \mathbb{R}$ such that

$$\langle c, v\rangle \geq \beta \geq \langle d, v\rangle$$

for all $(c, d) \in C \times D$. In other terms, the hyperplane $\langle \cdot, v\rangle^{-1}(\beta)$ separates C and D.

(ii) Given a linear subspace W and a compact, convex subset C such that $W \cap C = \emptyset$, there exists $v \in W^{\perp}$ (i.e. the orthogonal complement of W) and a real number $\beta > 0$ satisfying

$$\langle c, v\rangle \geq \beta$$

for all $c \in C$.

Hints for proving the first statement: 1. Show that it suffices to consider *closed* convex subsets by exhausting C and D by increasing sequences of compact, convex sets. 2. Recall why any non-empty closed set in V contains a vector of minimum length. 3. Show the statement for C and D closed.

2. The no-arbitrage assumption

In this exercise, we always consider a financial market with no arbitrage opportunities.

(i) Let $t_1 < \cdots < t_n < T$ be future times alias positive real numbers. We consider a market containing a bond with maturity T, face value $C > 0$ and coupon payments $c_1, \ldots, c_n > 0$ at times $t_1, \ldots, t_n$, respectively, as well as, for any $t > 0$, $B(0, t)$ alias a zero-coupon bond with maturity t. Please deduce the time-0-price of the coupon-paying bond as a function of those of the zero-coupon bonds!

(ii) Please show that the prices of American call and put options are greater than or equal to the prices of European call and put options, respectively, if their maturities are the same!

(iii) We suppose now that our market has only two time points, namely 0 and 1. We consider two bonds with maturity 1: B_0 and B_1, B_0 having zero coupon, B_1 having a coupon payment of $c > 0$ at time $0+$. We suppose that

$$B_0(0-) = B_0(1) > 0.$$

Date: October 16, 2019.

[1]... in particular without appealing to the Hahn-Banach theorem.

How does the coupon payment at time 0 alter the price of B_1, i.e. what is the relation between $B_1(0-)$ and $B_1(1)$?[2]

(iv) Please show the following more general form of the comparison lemma, cf. lemma 1.6 in the lecture notes. Let $\mathcal{T} \subset [0,T]$ be a (finite) set of time points containing 0 and $T > 0$. Given a market of financial securities (say: stock, bonds, options) which are all perfectly traded in the sense that they all have prices at any instant in $\mathcal{T}$ at which they can be bought and sold in arbitrary quantity. Then, if X denotes the price process of some self-financing portfolio including some (of course, finite) number of those securities, $X(T) \geq 0$ implies $X \geq 0$. The same holds with the $\geq$-signs being replaced by equality signs.

3. Financial engineering

In this exercise we consider a frictionless financial market from a point of view which is as model-free as possible, in the sense of a platform where some stock can be bought and sold at one unique price at each time instant in arbitrary quantity, idem for any option written on it. Time may be imagined as some subset of the real interval $[0,T]$ containing the start point 0 and the maturity $T > 0$, prices as positive real numbers. We assume that the market contains a zero-coupon bond B.

(i) An oil company issues the following bond. The holder receives no interest. At the bond's maturity T the company promises to pay \$1,000 plus an additional amount based on the price of oil at that time. The additional amount is equal to the product of 170 and the excess (if any) of the price of a barrel of oil at maturity over \$25. The maximum additional amount paid is \$2,550 (which corresponds to a price of \$40 per barrel). Please show that the bond is a combination of a zero-coupon bond, and one long and one short position in call options on oil. Please determine a) the size of the positions, and b) the face value and the strikes, respectively.

(ii) Given a call and a put option on the same underlying asset, having the same maturities and strikes, what is the condition on the price of the underlying at some time $t \leq T$ for that both options have the same price then?

(iii) A forward contract on an underlying asset S, with maturity T and strike K is an agreement between two parties that one of them is obliged to sell to the other one the asset S for the price K at time T, whatever the market price at time T may be. Please give the payoff function of a forward contract! Discuss the difference to a European option! Then, supposing that forward contract is about to be issued at a price of zero at time 0, please give the strike K which has to be chosen such that there are no arbitrage opportunities in the market!

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[2] The − and + signs after the time points could be skipped from a formal point of view, but have been used in order to insist on our convention assuming that the any individual buying a bond at instant t is already in its possession at this same instant t, and, therefore, – instantly – receives all coupon or dividend payments due at t.

[3] The strings preceding "@" have to be replaced by the corresponding family names. Replace the symbol eszett by "ss".