

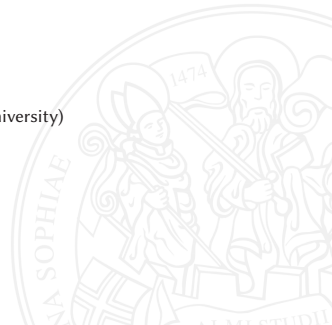
Option Pricing under Jump Uncertainty

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Joint work with **Olaf Menkens** (Dublin City University)

Presented at the
University of Bielefeld
June 20, 2018



Outline

- (1) Option Pricing under Jump Uncertainty: Model and Problem
- (2) Constrained BSDE and PDE Characterizations
- (4) Explicit Solutions and Numerical Results



Model and Problem Formulation



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$$P^i(\tau) = (1 + \beta_i)P^i(\tau-), \quad i = 1, \dots, n.$$



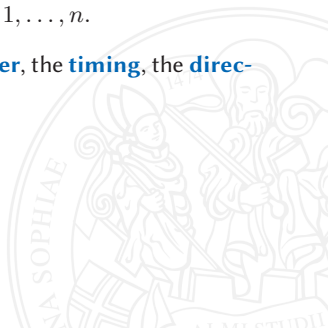
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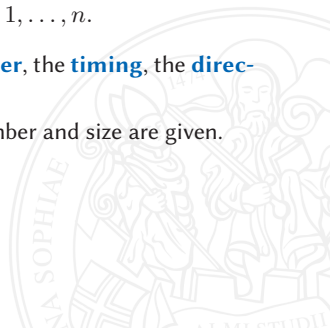
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Literature: Hua/Wilmott (1997), Mönnig (2012), Menkens (2016).

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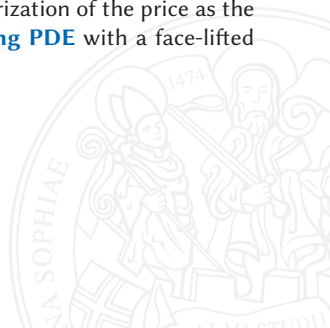
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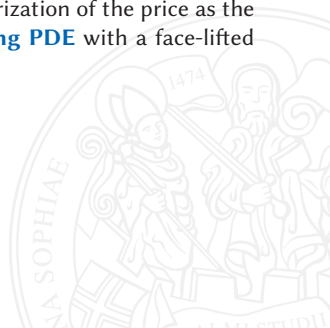
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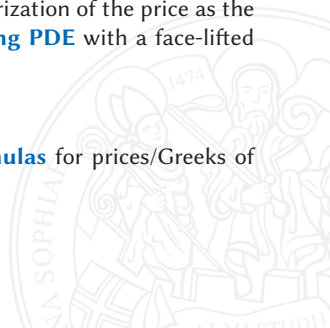
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- In the sister paper Menkens (2016): **Explicit formulas** for prices/Greeks of vanilla calls and puts.



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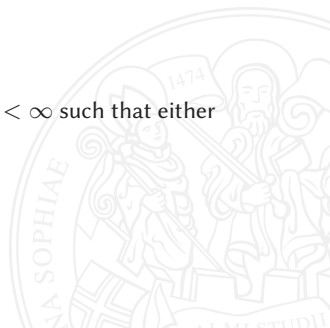
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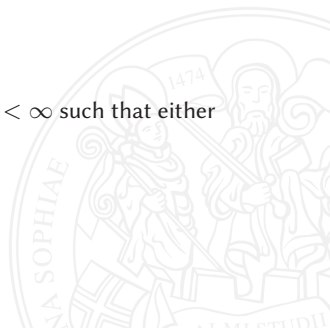
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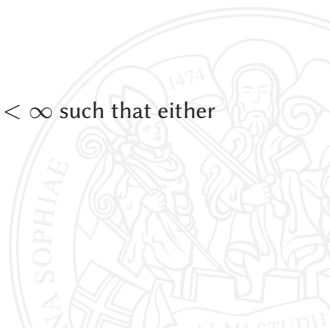
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Fair Price in the Absence of Jumps

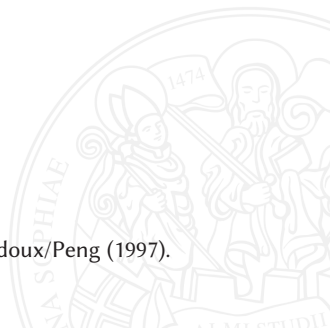
The **fair price** $\mathcal{V}_0(\tau; \xi)$ of ξ at time τ is given by

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where (Y, Z, K) is the **minimal solution of the BSDE**

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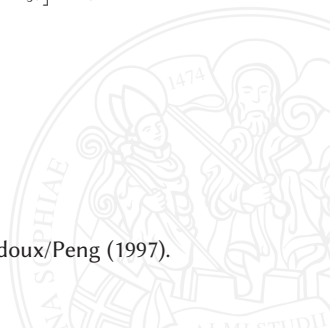
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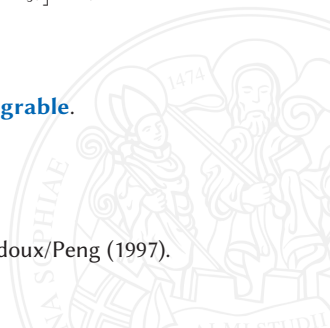
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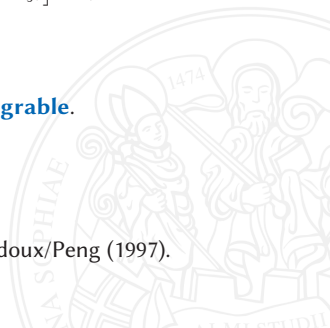
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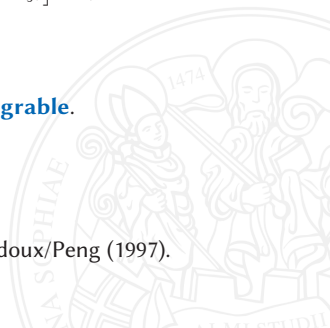
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Wealth Dynamics in the Presence of Jumps

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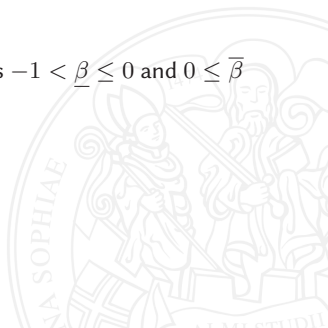
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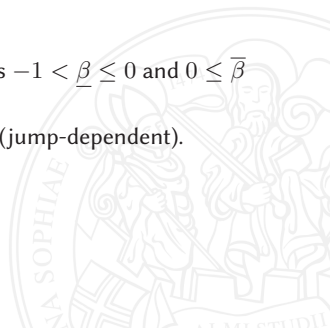
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Extensions: At most n **jumps, changing market parameters** after a jump (possibly jump-dependent).

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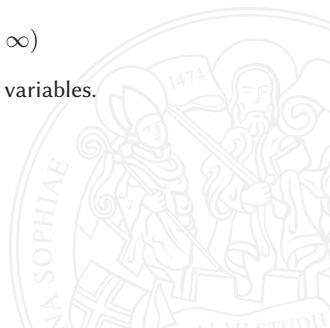
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We denote by $V_0(t, \beta, \omega)$ the **jump-free price** of ξ at time t in state ω , **conditional on the jump** (t, β) in the underlying.

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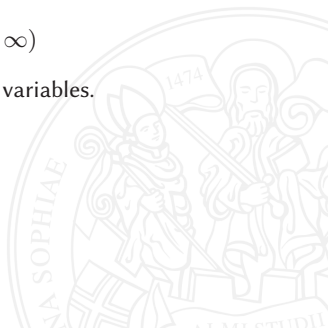
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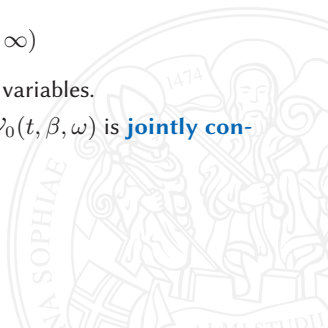
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Remark: Sufficient conditions for the last assumption are available. The proof is based on the Kolmogorov-Centsov continuity theorem.

The Jump Uncertainty Price

Definition: Jump Uncertainty Price

The **jump uncertainty price** \mathcal{V}_1 at time $t = 0$ is defined as

$\mathcal{V}_1 \triangleq \inf \left\{ x \geq 0 : \text{there exists } (\zeta^1, \{\zeta^\vartheta\}_\vartheta) \text{ such that} \right.$

$$X_t^{\zeta^1, \zeta^\vartheta, \vartheta} \geq \xi_t \text{ for all } t \in [0, T]$$

$\left. \text{and every jump scenario } \vartheta = (\tau, \beta) \right\}$.

In other words: The jump uncertainty price is the **smallest initial capital** required to **superhedge** the option in **every conceivable jump scenario** (including the no-jump scenario).

When is the Jump Uncertainty Price Finite?

At the time of a jump in the underlying, the **wealth jumps** as well:

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Proposition

Suppose that

$$\xi_E(P) \leq C(1 + \sum_{i=0}^n P_T^i)$$

in the European case or

$$\xi_A(t, P_t) \leq C(1 + \sum_{i=0}^n P_t^i)$$

in the American case. Then there exists at least one superhedging strategy and the jump uncertainty price is finite.

Constrained BSDE and PDE Characterizations



Characterization of the Set of Superhedging Strategies

The Main Idea

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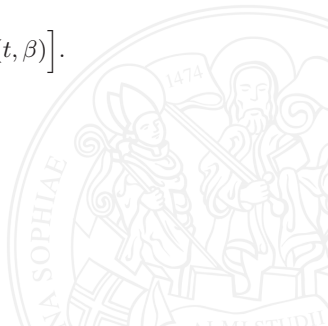
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Mathematically, this means

$$H(\tau, X_{\tau-}^{x, \zeta^1}, \zeta_\tau^1) \geq 0 \quad \text{for all stopping times } \tau, \quad (\text{JC})$$

where

$$H(t, x, z) \triangleq \inf_{\beta \in [\underline{\beta}_t, \bar{\beta}_t]} [x + \beta^\top z - \mathcal{V}_0(t, \beta)].$$



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Theorem

There exists a superhedging strategy for ξ if and only if there exist x, ζ^1 satisfying the jump constraint (JC).

BSDE Characterization of the Jump Uncertainty Price

Given the characterization of the set of superhedging strategies, we now "merely" have to find (x, ζ^1) satisfying (JC) requiring the least initial wealth.



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Theorem (BSDE Characterization)

Suppose that there exists at least one superhedging strategy for ξ . Then $\mathcal{V}_1 = Y_0$, where (Y, Z, C) is the minimal solution of the BSDE

$$dY_t = Z_t \theta_t dt + Z_t dW_t - dC_t, \quad t \in [0, T], \quad Y_T = \xi_T$$

satisfying the constraint

$$H(t, Y_t, \sigma_t^{-\top} Z_t) \geq 0 \quad \text{for all } t \in [0, T] \text{ almost surely.}$$

A corresponding pre-jump superhedging strategy is given by $\zeta^1 \triangleq \sigma^{-\top} Z$.

The **proof** proceeds in several steps:

- (1) Show that the existence of a superhedging strategy implies the existence of at least one solution of the BSDE satisfying the jump constraint (JC).



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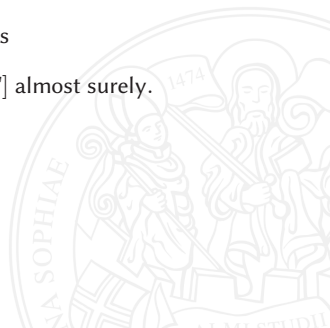
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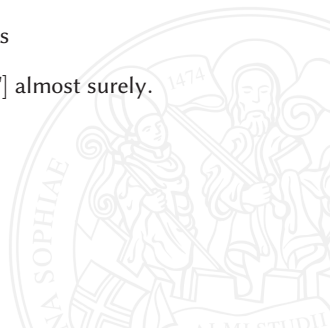
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The Markovian Market Model

Suppose now that we are in a **Markovian setting**, i.e. $\sigma, \theta, \underline{\beta}, \bar{\beta}, \xi$ are given in terms of deterministic functions of the prices of the underlyings.



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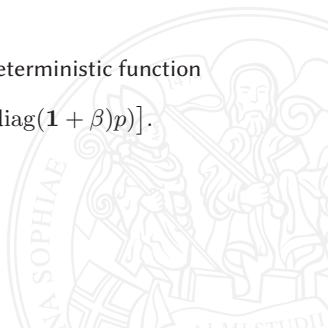
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Literature: Kharroubi/Ma/Pham/Zhang (2010), Peng/Xu (2013), Bouchard (2002), Bouchard/Elie/Touzi (2009).

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Suppose that ξ satisfies a linear growth condition. Then \mathcal{V}_1 is the unique continuous viscosity solution of

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Proof: Only interesting if you enjoy viscosity solutions as much as I do...

Explicit Solutions and Numerical Results



Explicit Solutions for European Vanilla Options

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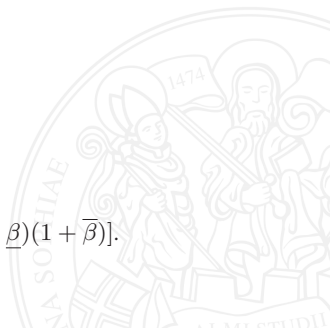


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Remark: The adjusted strike L is given by $L = K / [(1 + \underline{\beta})(1 + \overline{\beta})]$.

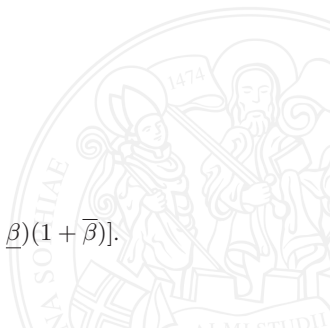


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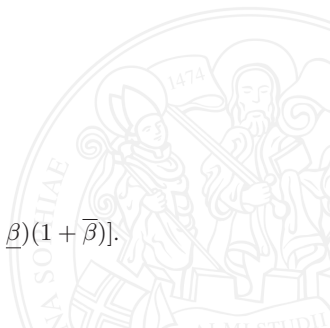


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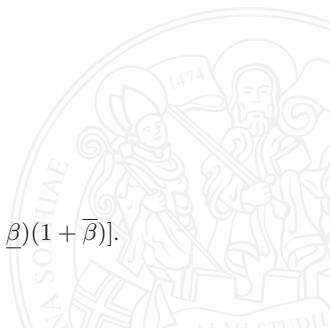
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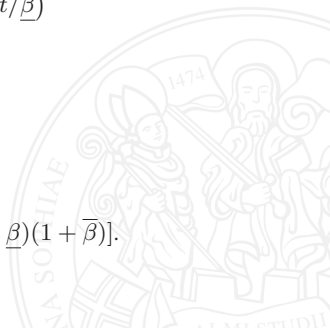
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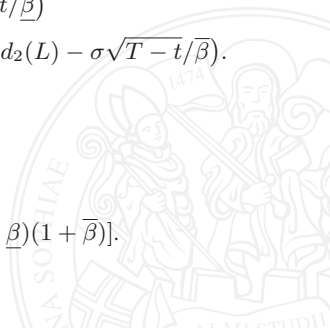
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Interestingly: The uncertainty premium $\mathcal{V}_1^{\text{EP}} - \mathcal{V}_0^{\text{EP}}$ at time $t < T$ is simply the (jump-free) Black-Scholes price of the uncertainty premium $\mathcal{V}_1^{\text{EP}}(T-, p) - [K - p]^+$ at terminal time.

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Well known: In the absence of jump uncertainty, the price V_0^{AC} of the American call coincides with the price of the European call V_0^{EC} .



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This implies: The corresponding jump uncertainty prices $\mathcal{V}_1^{\text{AC}}$ and $\mathcal{V}_1^{\text{EC}}$ satisfy the same PDE, i.e.

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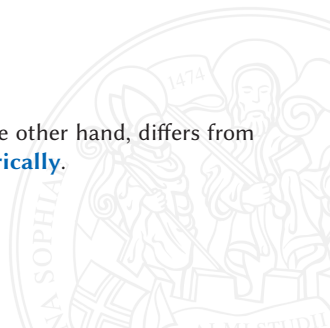
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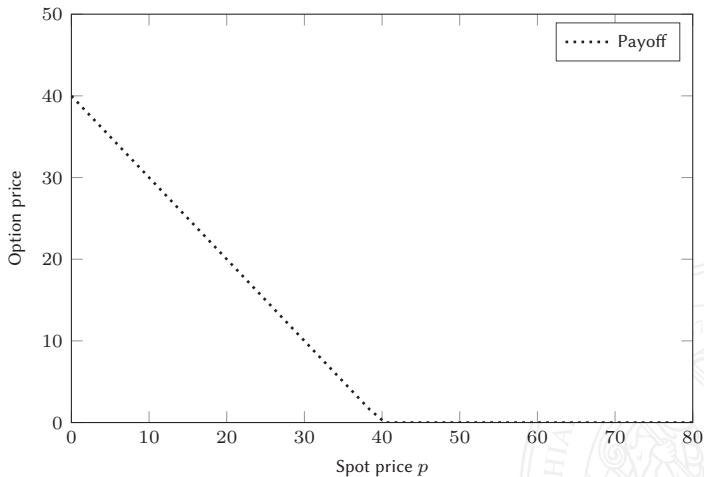
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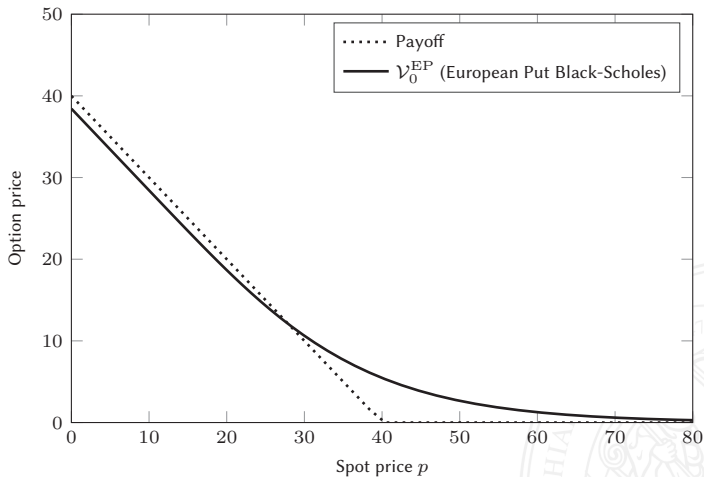
The jump uncertainty price of the **American put**, on the other hand, differs from that of the European put and has to be computed **numerically**.



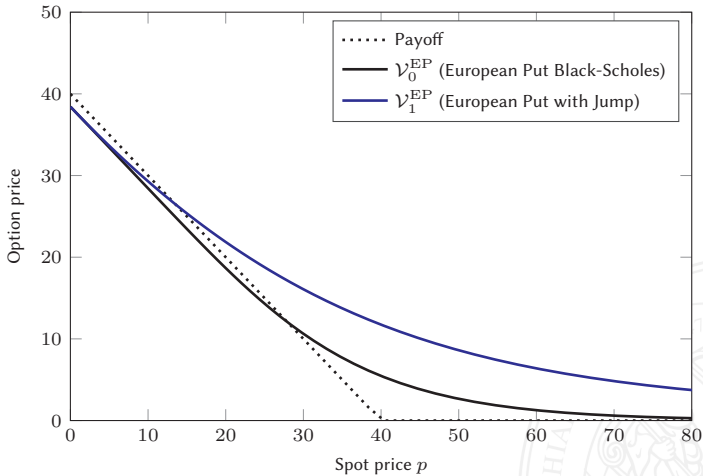
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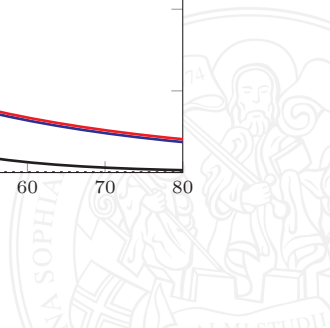
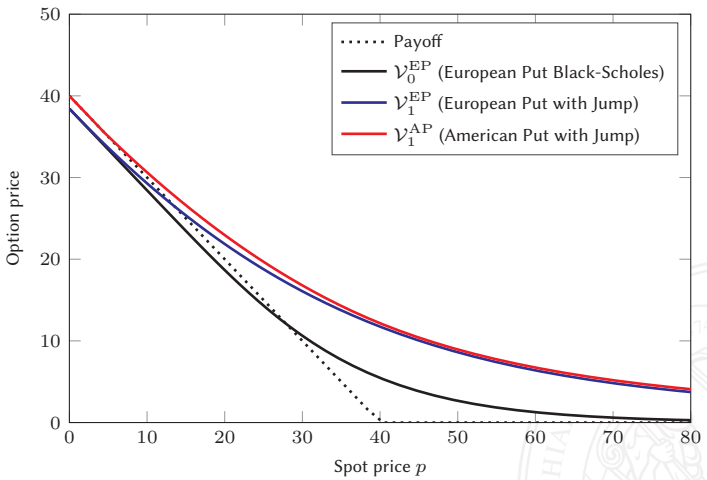
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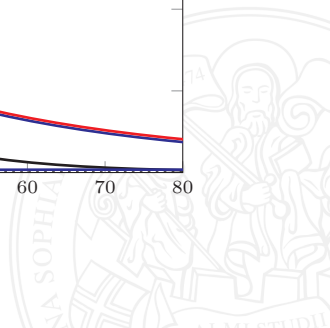
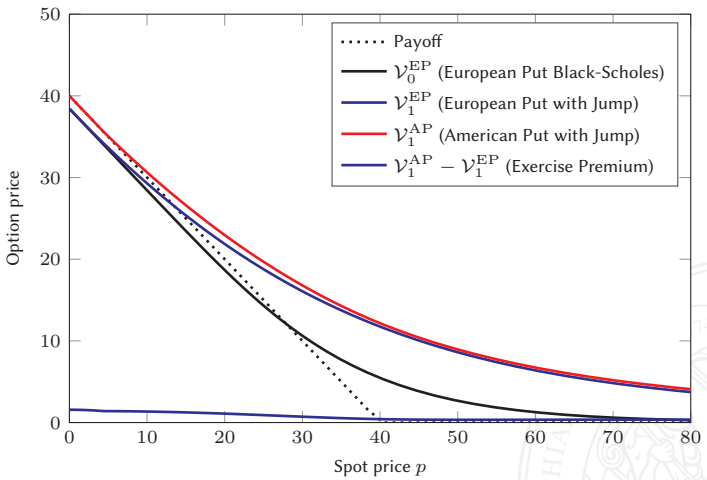
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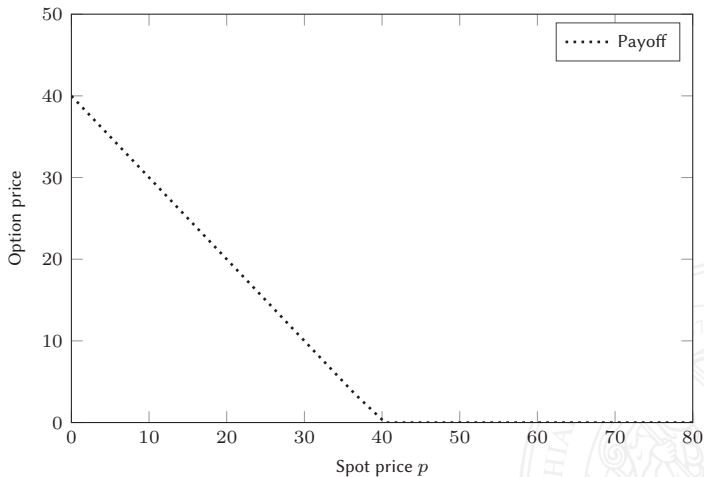
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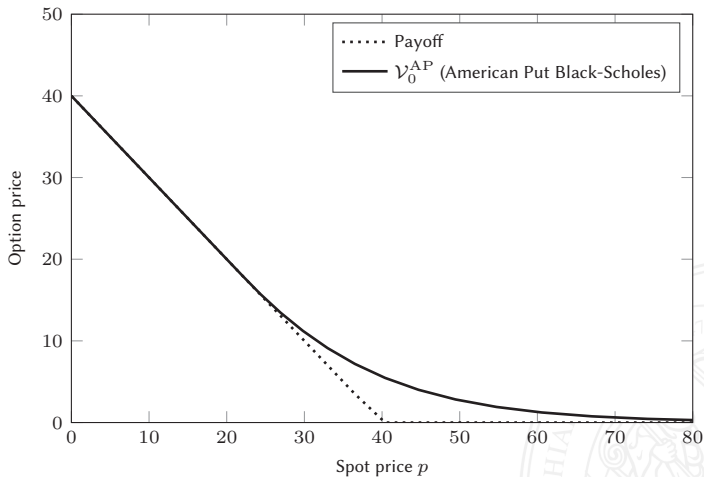
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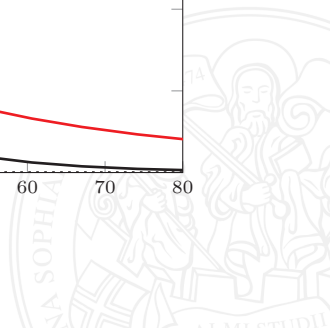
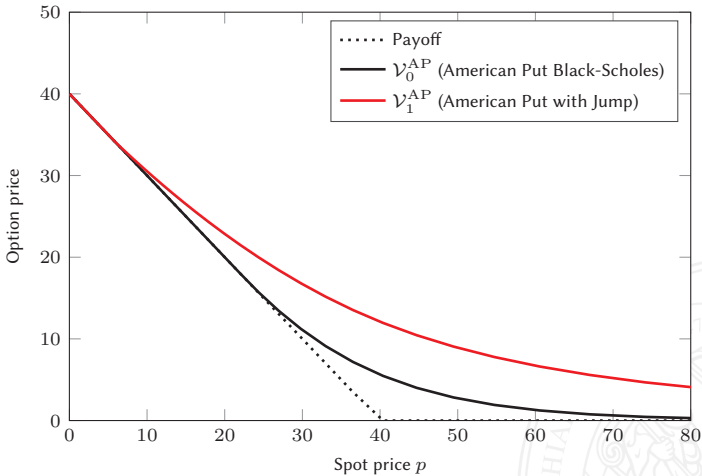
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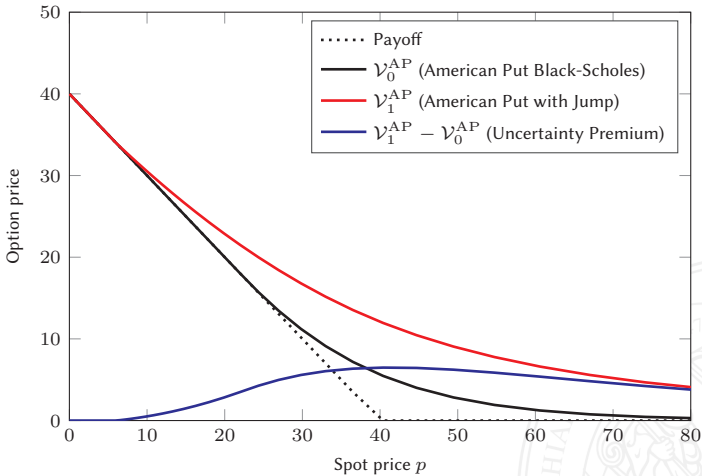
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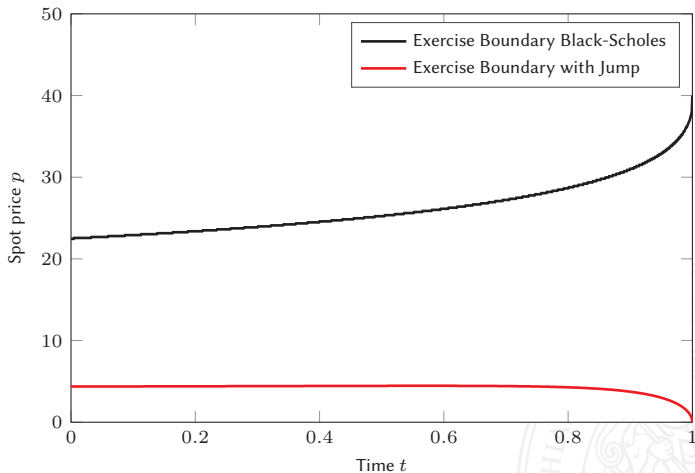
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Exercise Boundaries for the American Put



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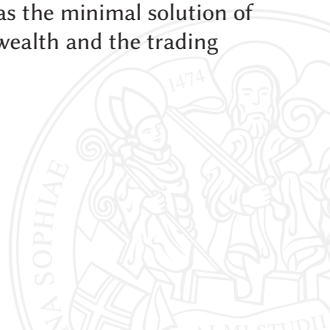
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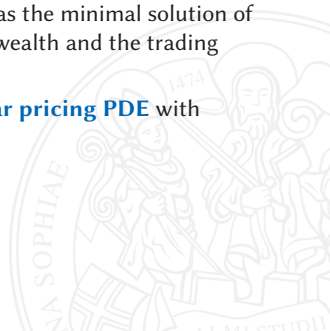
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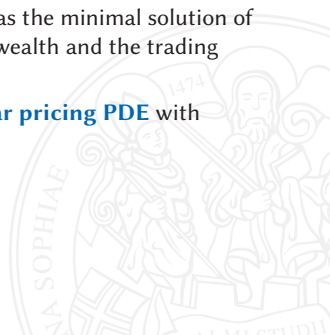
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