Universität Trier

Utility Maximization with Constant Costs

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Joint work with **Sören Christensen** (University of Hamburg).

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Outline

(1) The Problem: Utility Maximization with Constant Costs

(2) (Dis-)Continuity of the Value Function

(3) Construction of Optimal Strategies

(4) Conclusion and Outlook

Utility Maximization with Constant Costs

Portfolio Optimization with Transaction Costs

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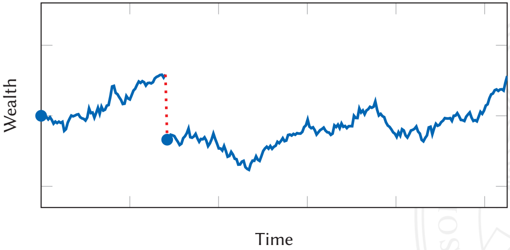
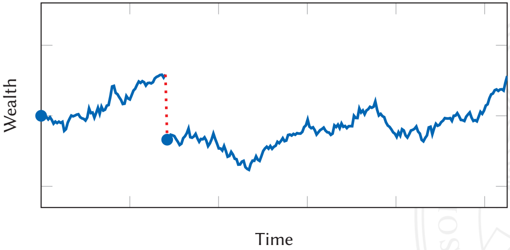


Portfolio Optimization with Transaction Costs

Our aim is to solve a portfolio optimization problem which involves both **proportional** and **constant** transaction costs.

Trading strategies are modeled as **impulse controls** $\{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$, where

- τ_k denotes the **time** of the kth transaction (stopping time),
- Δ_k denotes the **volume** of the kth transaction (random variable).



The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\mathrm{d}X_1(t) = rX_1(t)\mathrm{d}t, \qquad t \in [\tau^k, \tau^{k+1}),$$
$$\mathrm{d}X_2(t) = \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), \qquad t \in [\tau^k, \tau^{k+1}),$$

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X_1(\tau_k) &= X_1(\tau_k-) - \Delta_k\\
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We **prohibit short selling** of the stock. A portfolio $x \in \mathbb{R} \times [0,\infty)$ is **solvent** if it has a positive liquidation value $\mathrm{L}(x)$, i.e.,

$$\mathrm{L}(x) \triangleq x_1 + \left(x_2 - \gamma x_2 - C\right)^+ > 0.$$

The set $\mathcal{S} \subset \mathbb{R} \times [0,\infty)$ of solvent portfolios is called the **solvency region**.

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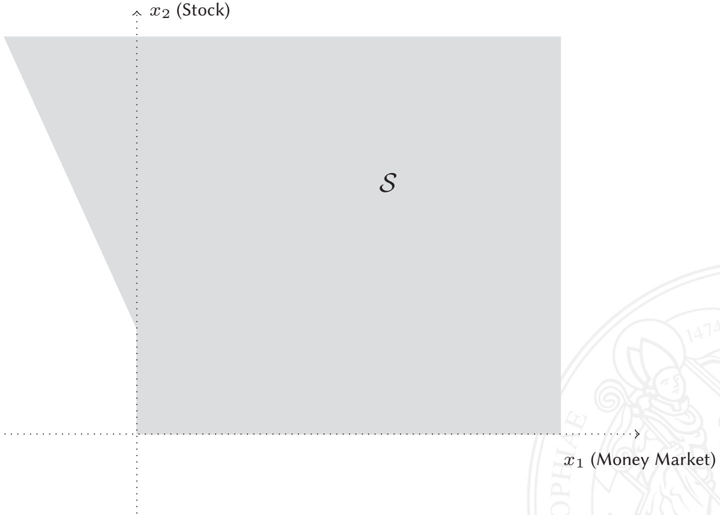

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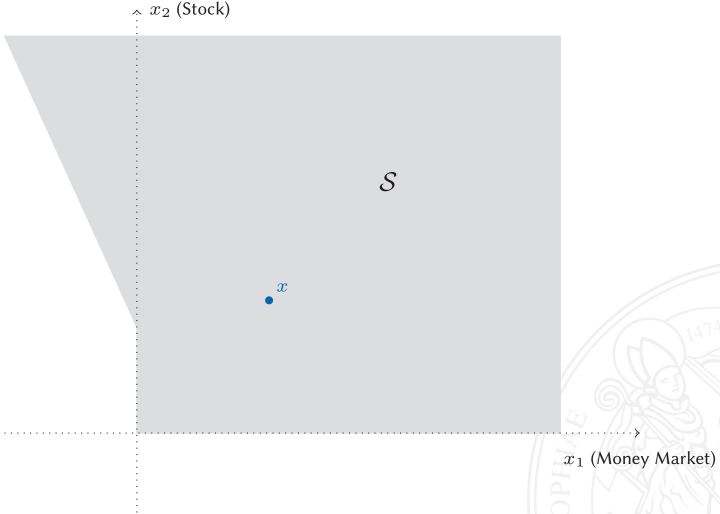

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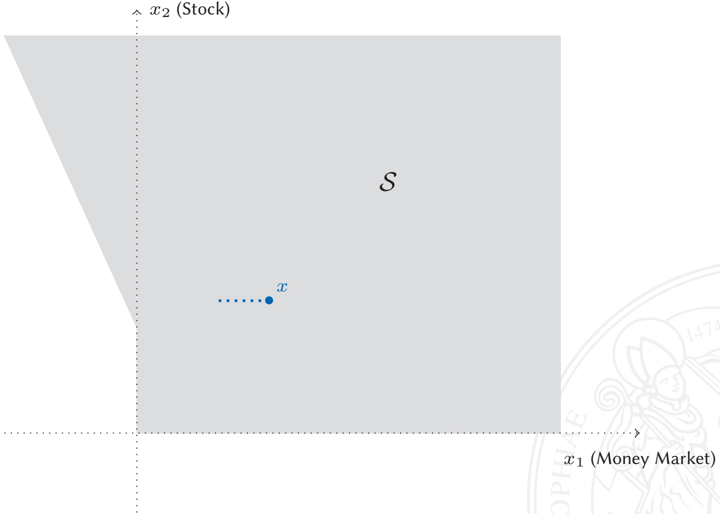

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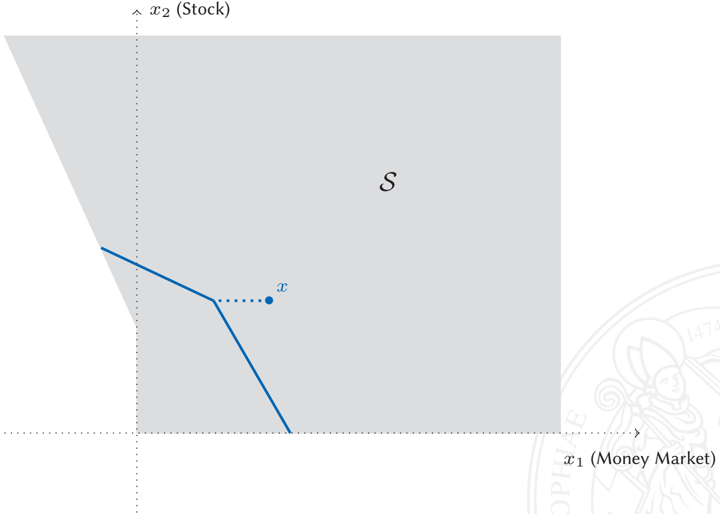

The Optimization Criterion

Now fix a **utility function** $U : \mathbb{R}_+ \to \mathbb{R}$ such that

- U is strictly increasing, continuous, and concave,
- U is lower bounded; without loss of generality $U(0) = 0$,
- U satisfies $U(\ell) \le M(1 + |\ell|^p)$ for some $M > 0$, $p \in (0, 1)$.

The objective is to **maximize expected utility of terminal wealth**, i.e.

$$\mathcal{V}(t, x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[U\Big(\mathrm{L}\big(X^{\Lambda}_{t,x}(T)\big)\Big)\Big],$$

where $\mathcal{A}(t, x)$ denotes the set of **admissible strategies** Λ for the initial state (t, x).

The Quasi-Variational Inequalities

The value function $\mathcal{V}$ is expected to be linked to the following **quasi-variational inequalities** (QVIs):

$$\min\Big\{-\partial_t \mathcal{V}(t,x) - \mathcal{L}\mathcal{V}(t,x), \mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)\Big\} = 0, \quad (t,x) \in [0,T) \times \mathcal{S},$$

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where $\mathcal{L}$ denotes the **infinitesimal generator** of the uncontrolled portfolio process given by

$$\mathcal{L}\mathcal{V}(t,x) \triangleq r x_1 \partial_{x_1} \mathcal{V}(t,x) + \mu x_2 \partial_{x_2} \mathcal{V}(t,x) + \frac{1}{2}\sigma^2 x_2^2 \partial^2_{x_2,x_2} \mathcal{V}(t,x),$$

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and $\mathcal{M}$ is the **maximum operator** given by

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta} \mathcal{V}(t, x_1 - \Delta - \gamma|\Delta| - C, x_2 + \Delta).$$

Related Literature

In the existing literature, there are three different approaches:

- **Classical Verification**: Eastham and Hastings (1988), Korn (1998), Bielecki and Pliska (2000), Liu (2004).
- **Viscosity Solutions**: Oksendal and Sulem (2002), Altarovici, Reppen, and Soner (2016)
- **Small Cost Asymptotics**: Korn (1998), Korn and Laue (2002), Altarovici, Muhle-Karbe, and Soner (2015), Feodoria and Kallsen (2016), Altarovici, Reppen, and Soner (2016).

Note: To the best of our knowledge, no existence result for **optimal strategies** and no **uniqueness** result for the value function is known.

(Dis-)Continuity of the Value Function

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Hence: If $\mathcal{V}_* \geq \mathcal{V}^*$ on the boundary, then $\mathcal{V}_* \geq \mathcal{V}^*$ everywhere, i.e.

$$\mathcal{V} = \mathcal{V}_* = \mathcal{V}^* \text{ is continuous.}$$

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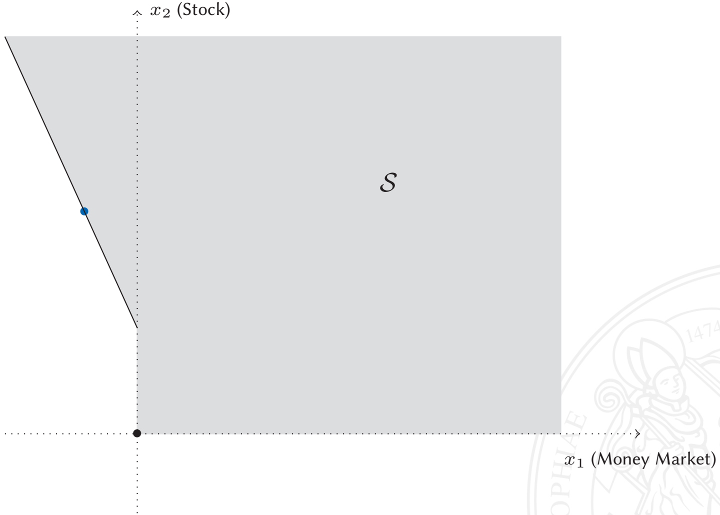

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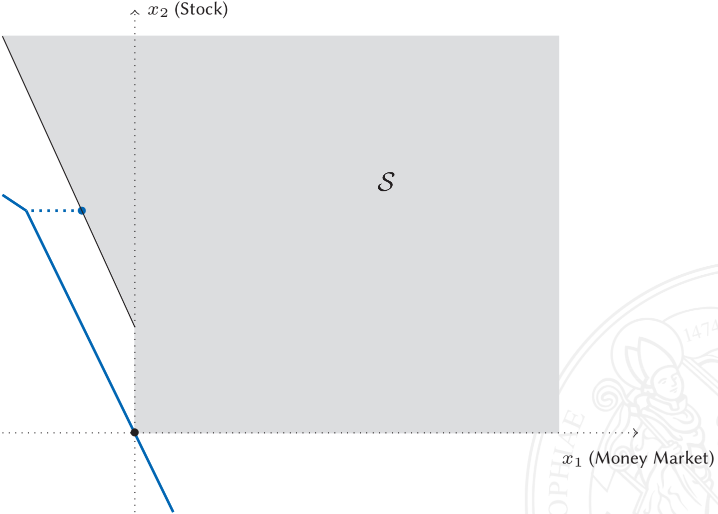

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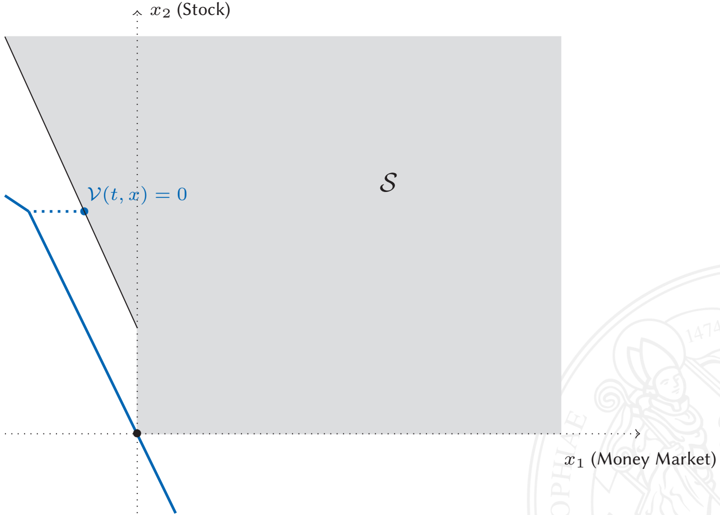

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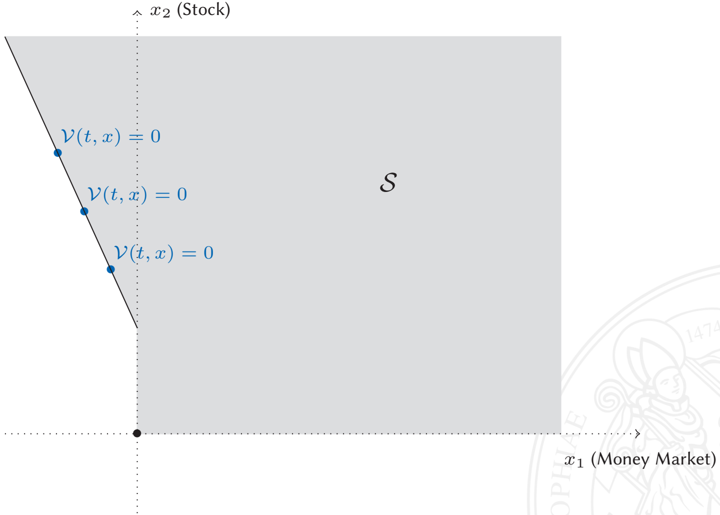

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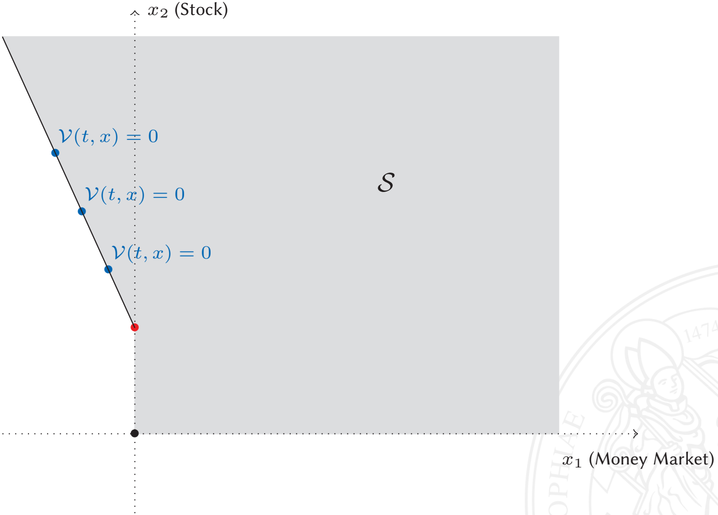

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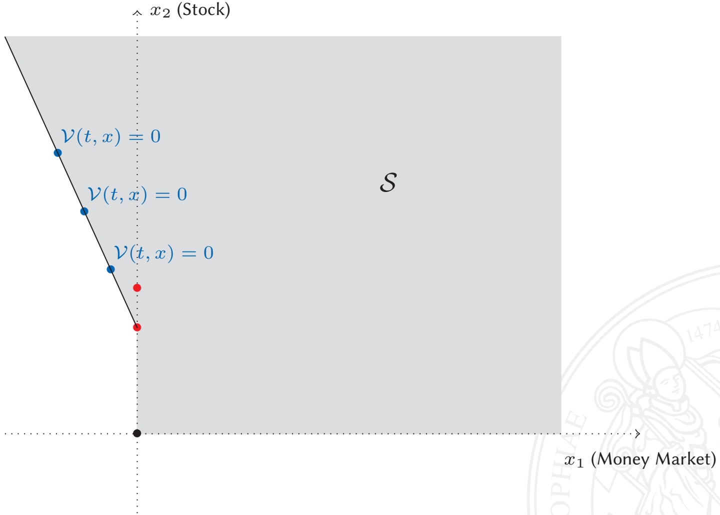

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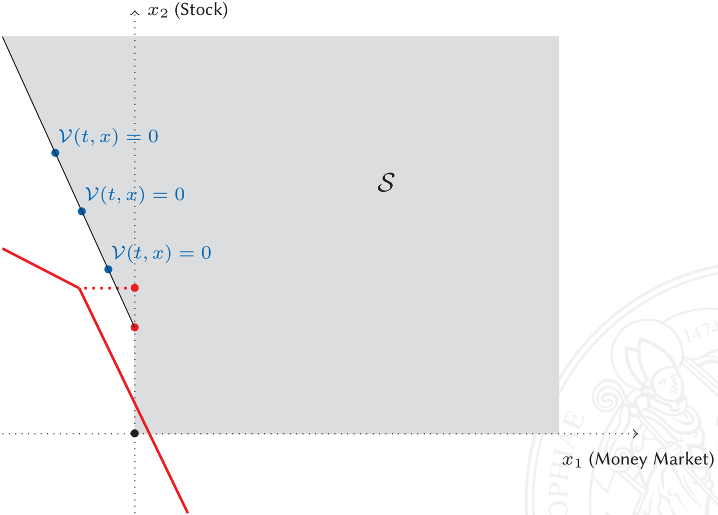

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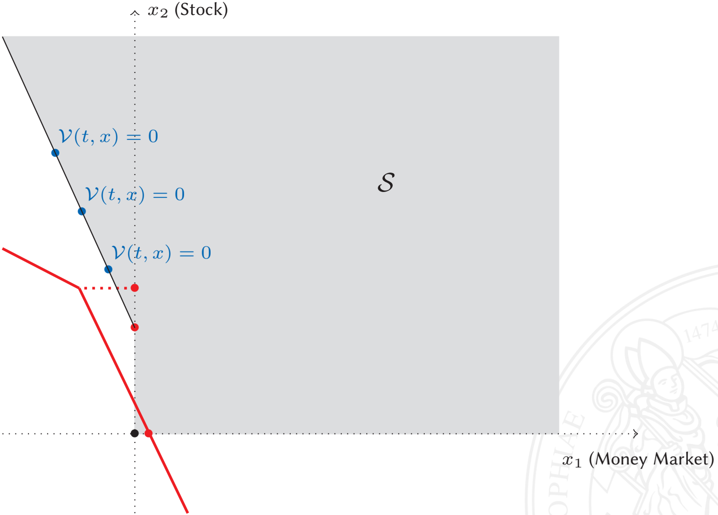

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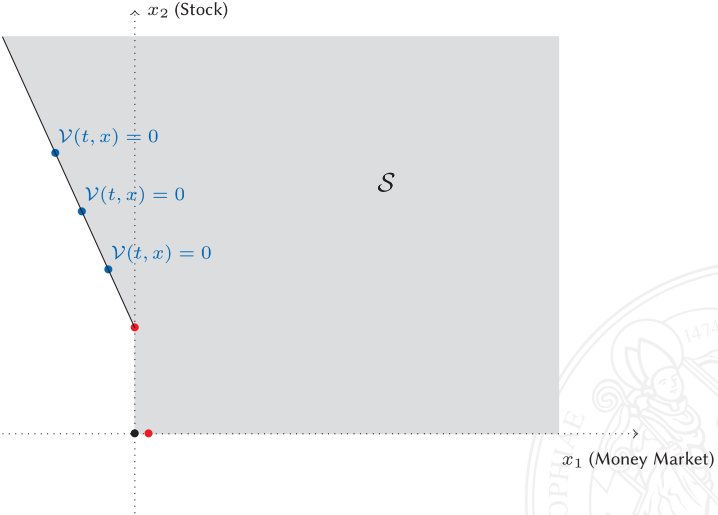

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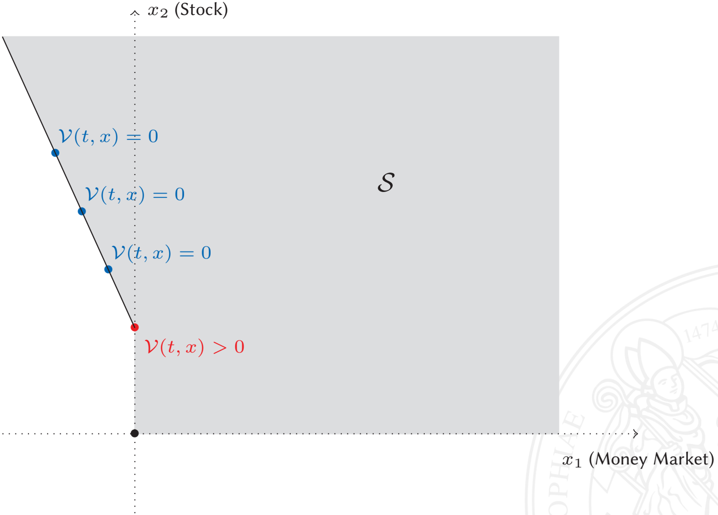

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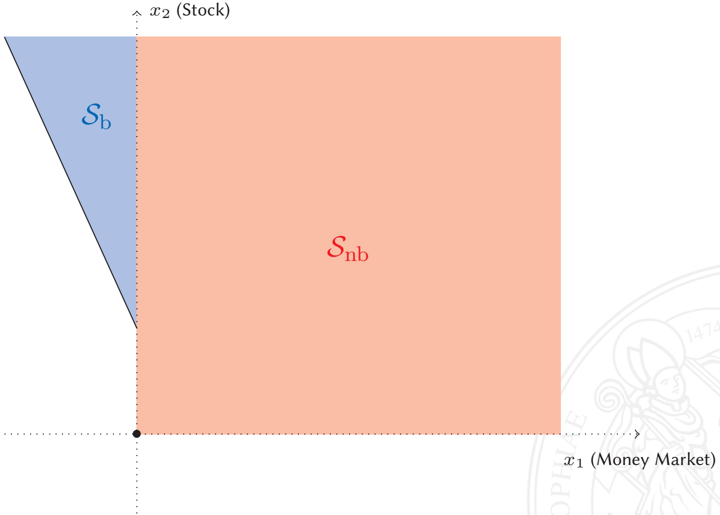
Difficulty: The QVIs have a **non-local** term: $\mathcal{V}(t, x) - \mathcal{M}\mathcal{V}(t, x)$.

Localizing the Solvency Region



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This implies that the value function $\mathcal{V}$ is

- **continuous** if restricted to $[0,T] \times \mathcal{S}_{\mathrm{b}}$,

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The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the QVIs,

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Next step: Try to construct **optimal strategies**.

Construction of Optimal Strategies

A Candidate Optimal Strategy

Recall that the **maximum operator** $\mathcal{M}$ was defined as

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta} \mathcal{V}\big(t, x_1 - \Delta - \gamma|\Delta| - C, x_2 + \Delta)\big).$$

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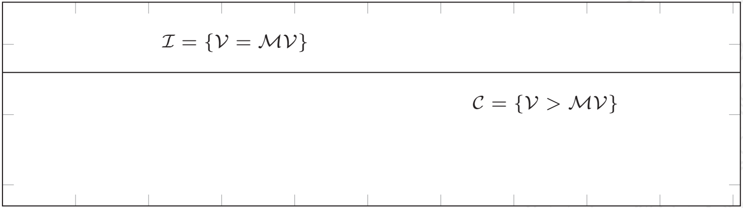

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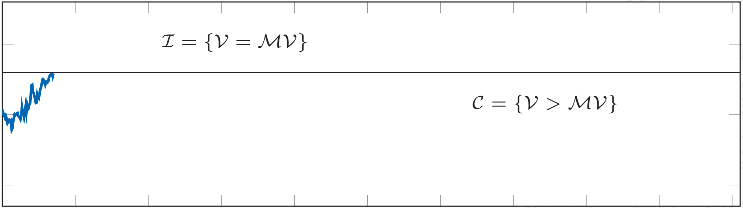

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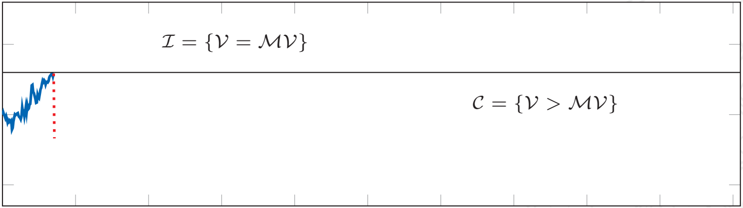

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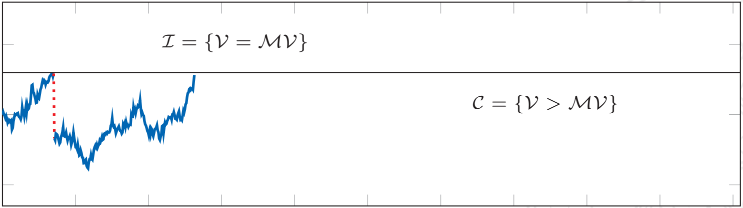

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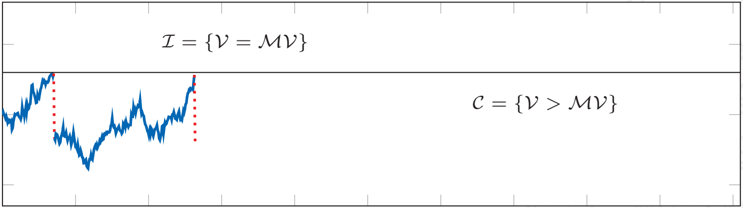

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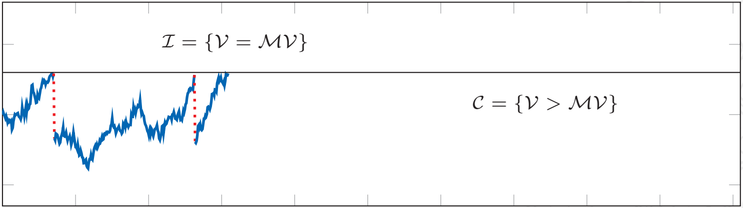

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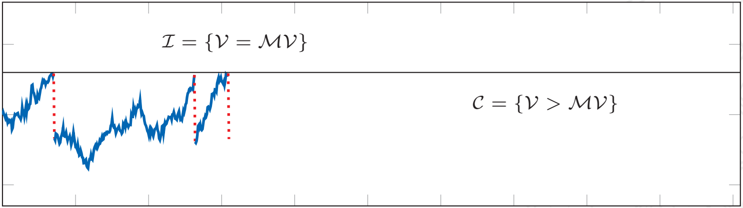

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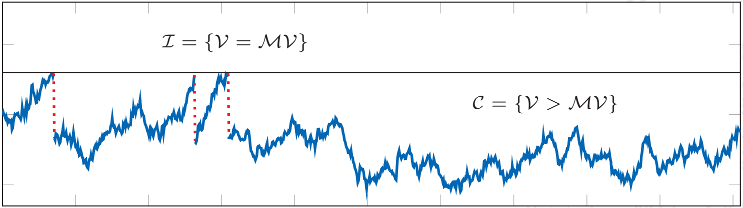

An Implicit Optimal Stopping Problem

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Attention: $\mathcal{V}$ is **not globally** lower semicontinuous. However, the uncontrolled state process never crosses the x_2-axis. So localization should work.

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Proof: Iteratively solve the implicit optimal stopping problem. Uses classical optimal stopping techniques and that $\mathcal{V}$ is the pointwise minimum of $\mathbb{H}$.

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- **Numerical results** are work in progress.