Universität Trier

Non-Smooth Verification for Impulse Control Problems

Christoph Belak

Department IV – Mathematics
University of Trier
Germany

Joint work with **Sören Christensen** (Hamburg) and **Frank Seifried** (Trier).

Optimal Stopping in Complex Environments Workshop
Bielefeld, December 18, 2017

Outline

(1) A New Approach to Impulse Control Problems

(2) Utility Maximization with Constant Costs

(3) (Dis-)Continuity of the Value Function

Stochastic Impulse Control Problems

Based on:
Belak, C., Christensen, S., and Seifried, F. T.: **A general verification result for stochastic impulse control problems**, *SIAM Journal on Control and Optimization*, Vol. 55, No. 2, pp. 627-649, 2017.

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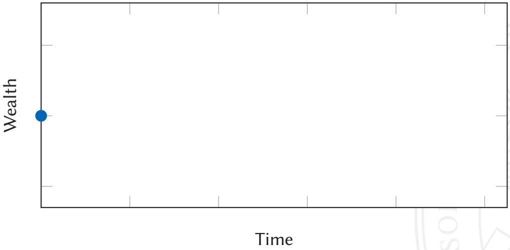

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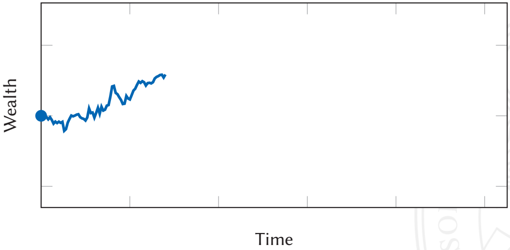

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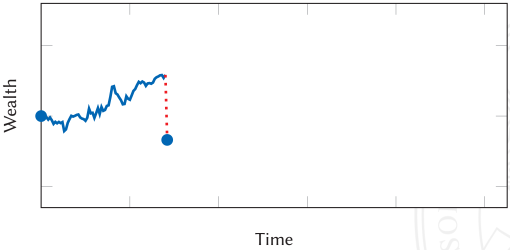

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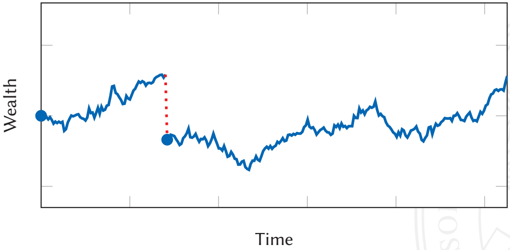

The General Impulse Control Problem

Consider an $\mathbb{R}^n$-valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} \mathrm{d}X(t) &= \mu(X(t))\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), && t \in [\tau_k, \tau_{k+1}), \\ X(\tau_k) &= \Gamma(X(\tau_k -), \Delta_k), \end{aligned}$$

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- the stopping times τ_k are **increasing** and **do not accumulate** in that

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The **objective** is to find a maximizer of

$$\mathcal{V}(t,x) \triangleq \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[g\big(X^\Lambda_{t,x}(T)\big)\Big].$$

A Candidate Optimal Control

Define the so-called **maximum operator** $\mathcal{M}$ via

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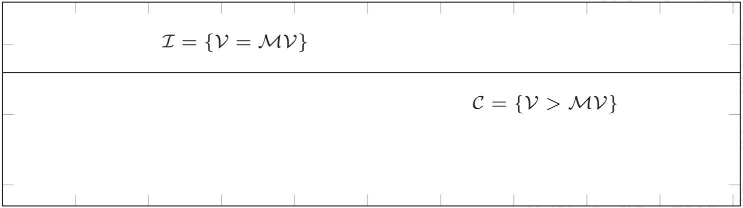

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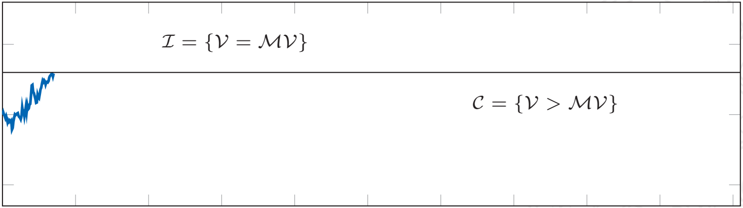

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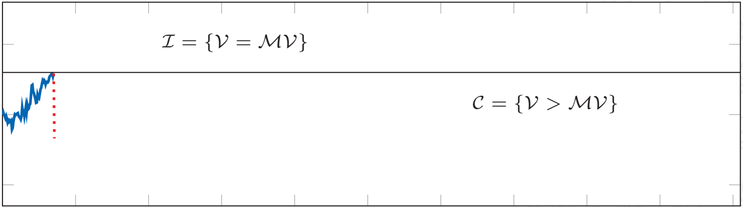

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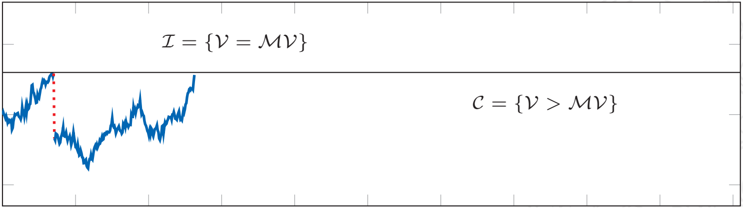

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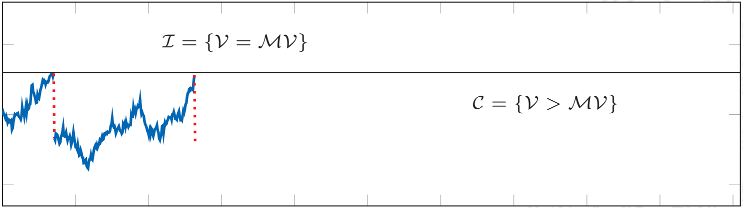

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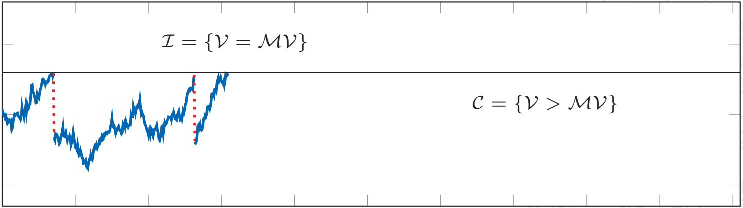

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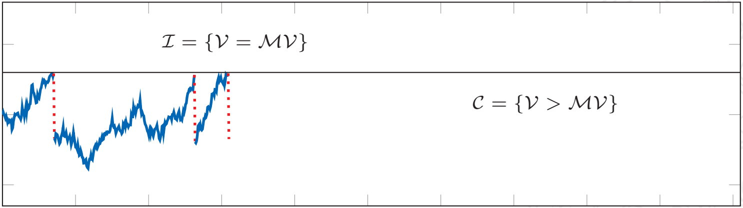

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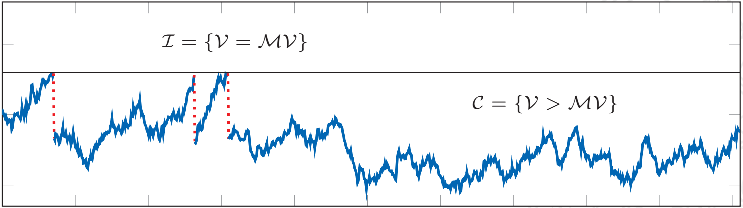

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Remark: Under standard assumptions, $\mathcal{M}$ **preserves semicontinuity**, i.e. $G = \mathcal{M}\mathbb{V}$ is upper semicontinuous if $\mathbb{V}$ is upper semicontinuous. That is, continuity of $\mathbb{V}$ should suffice to solve the problem!

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Proof: Relies mainly on classical optimal stopping techniques, i.e. works in quite general settings.

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- How well does this approach work in **practice**, i.e. for specific models?

Utility Maximization with Constant Costs

Based on:
Belak, C. and Christensen, S.: **Utility maximization in a factor model with constant and proportional costs**, Preprint, available on SSRN, 2017.

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\mathrm{d}X_1(t) = rX_1(t)\mathrm{d}t, \qquad t \in [\tau^k, \tau^{k+1}),$$

$$\mathrm{d}X_2(t) = \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), \qquad t \in [\tau^k, \tau^{k+1}),$$

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where $\gamma \in (0,1)$ (**proportional cost**) and $C > 0$ (**constant cost**).

We **prohibit short selling** of the stock. A portfolio $x \in \mathbb{R} \times [0,\infty)$ is **solvent** if it has a positive liquidation value $\mathrm{L}(x)$, i.e.,

$$\mathrm{L}(x) \triangleq x_1 + \big(x_2 - \gamma x_2 - C\big)^+ > 0.$$

The set $\mathcal{S} \subset \mathbb{R} \times [0,\infty)$ of solvent portfolios is called the **solvency region**.

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The Optimization Criterion

Now fix a **utility function** $U : \mathbb{R}_+ \to \mathbb{R}$ such that

- U is strictly increasing, continuous, and concave,
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The objective is to **maximize expected utility of terminal wealth**, i.e.

$$\mathcal{V}(t, x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[U\Big(\mathrm{L}\big(X^{\Lambda}_{t,x}(T)\big)\Big)\Big],$$

where $\mathcal{A}(t, x)$ denotes the set of **admissible strategies** Λ for the initial state (t, x).

Additional Difficulties in this Model

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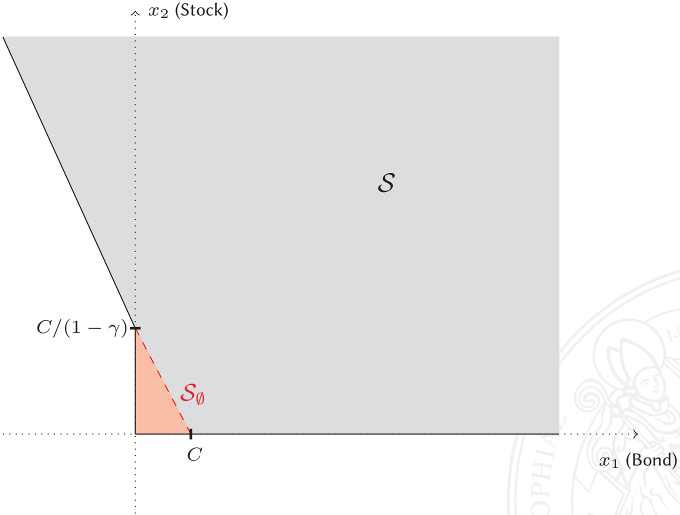
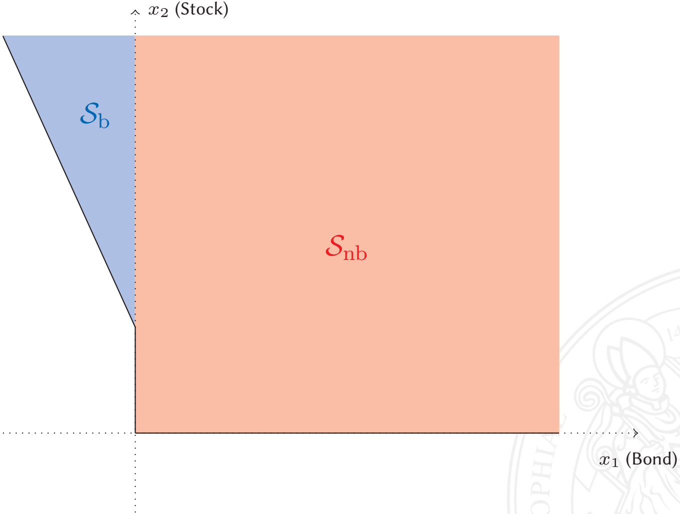
- ... the value function is **discontinuous**!

(Dis-)Continuity of the Value Function

The Solvency Region



Localizing the Solvency Region



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For any $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u° and $u_\circ$ the upper and lower semicontinuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_\circ$, where $\circ \in \{\mathrm{b}, \mathrm{nb}\}$.

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This implies that any viscosity solution with **piecewise continuous** boundary condition is **continuous if restricted** to $[0,T] \times \mathcal{S}_\circ$, $\circ \in \{\mathrm{b}, \mathrm{nb}\}$. The viscosity solution need **not** be globally continuous.

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- With this, we are able to construct an **optimal trading strategy**.

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Thanks for your attention and enjoy the coffee break!

Optimal Trading Regions