Universität Trier

Utility Maximization with Constant Costs

Christoph Belak

Department IV – Mathematics
University of Trier
Germany

Joint work with **Sören Christensen** (Hamburg) and **Frank Seifried** (Trier).

Joint Risk & Stochastics and Financial Mathematics Seminar
London School of Economics, September 28, 2017

Outline

(1) A New Approach to Impulse Control Problems

(2) Utility Maximization with Constant Costs

(3) (Dis-)Continuity of the Value Function

Impulse Control Problems

Based on:

Belak, C., Christensen, S., and Seifried, F. T.: **A general verification result for stochastic impulse control problems**, *SIAM Journal on Control and Optimization*, Vol. 55, No. 2, pp. 627-649, 2017.

Portfolio Optimization with Transaction Costs

Our aim is to solve a portfolio optimization problem which involves both **proportional** and **constant** transaction costs.

Portfolio Optimization with Transaction Costs

Our aim is to solve a portfolio optimization problem which involves both **proportional** and **constant** transaction costs.

Trading strategies are modeled as **impulse controls** $\{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$, where

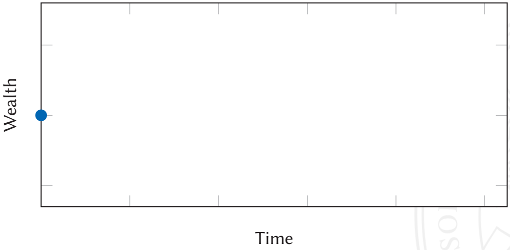
- τ_k denotes the **time** of the kth transaction (stopping time),
- Δ_k denotes the **volume** of the kth transaction (random variable).

Portfolio Optimization with Transaction Costs

Our aim is to solve a portfolio optimization problem which involves both **proportional** and **constant** transaction costs.

Trading strategies are modeled as **impulse controls** $\{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$, where

- τ_k denotes the **time** of the kth transaction (stopping time),
- Δ_k denotes the **volume** of the kth transaction (random variable).

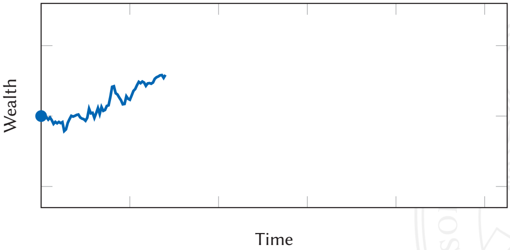


Portfolio Optimization with Transaction Costs

Our aim is to solve a portfolio optimization problem which involves both **proportional** and **constant** transaction costs.

Trading strategies are modeled as **impulse controls** $\{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$, where

- τ_k denotes the **time** of the kth transaction (stopping time),
- Δ_k denotes the **volume** of the kth transaction (random variable).

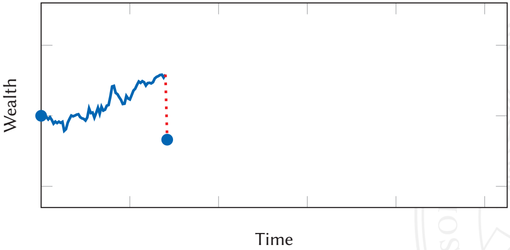


Portfolio Optimization with Transaction Costs

Our aim is to solve a portfolio optimization problem which involves both **proportional** and **constant** transaction costs.

Trading strategies are modeled as **impulse controls** $\{(\tau_k, \Delta_k)\}_{k \in \mathbb{N}}$, where

- τ_k denotes the **time** of the kth transaction (stopping time),
- Δ_k denotes the **volume** of the kth transaction (random variable).

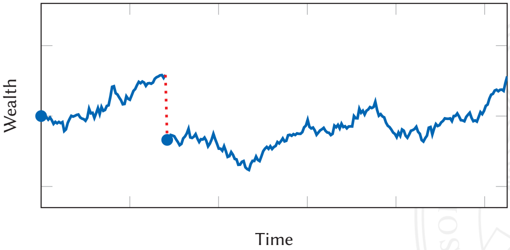


Portfolio Optimization with Transaction Costs

Our aim is to solve a portfolio optimization problem which involves both **proportional** and **constant** transaction costs.

Trading strategies are modeled as **impulse controls** $\{(\tau_k, \Delta_k)\}_{k \in \mathbb{N}}$, where

- τ_k denotes the **time** of the kth transaction (stopping time),
- Δ_k denotes the **volume** of the kth transaction (random variable).



The General Impulse Control Problem

Consider an $\mathbb{R}^n$-valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$ as follows:

$$
\begin{aligned}
\mathrm{d}X(t) &= \mu(X(t))\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \qquad t \in [\tau_k, \tau_{k+1}),\\
X(\tau_k) &= \Gamma(X(\tau_k-), \Delta_k),
\end{aligned}
$$

The General Impulse Control Problem

Consider an $\mathbb{R}^n$-valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} \mathrm{d}X(t) &= \mu(X(t))\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \qquad && t \in [\tau_k, \tau_{k+1}), \\ X(\tau_k) &= \Gamma(X(\tau_k-), \Delta_k), \end{aligned}$$

where

- the stopping times τ_k are **increasing** and **do not accumulate** in that

$$\mathbb{P}[\lim_{k\to\infty} \tau_k > T] = 1,$$

- the impulses Δ_k are chosen from a **state-dependent** set $Z(X(\tau_k-)) \subset \mathbb{R}^m$.

The General Impulse Control Problem

Consider an $\mathbb{R}^n$-valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} \mathrm{d}X(t) &= \mu(X(t))\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), && t \in [\tau_k, \tau_{k+1}), \\ X(\tau_k) &= \Gamma(X(\tau_k-), \Delta_k), \end{aligned}$$

where

- the stopping times τ_k are **increasing** and **do not accumulate** in that

$$\mathbb{P}[\lim_{k\to\infty} \tau_k > T] = 1,$$

- the impulses Δ_k are chosen from a **state-dependent** set $Z(X(\tau_k-)) \subset \mathbb{R}^m$.

The **objective** is to find a maximizer of

$$\mathcal{V}(t,x) = \sup_{\Lambda\in\mathcal{A}(t,x)} \mathbb{E}\Big[g\big(X^\Lambda_{t,x}(T)\big)\Big].$$

A Candidate Optimal Control

Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

A Candidate Optimal Control

Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

A Candidate Optimal Control

Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and

A Candidate Optimal Control

Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.

A Candidate Optimal Control

Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.

A Candidate Optimal Control

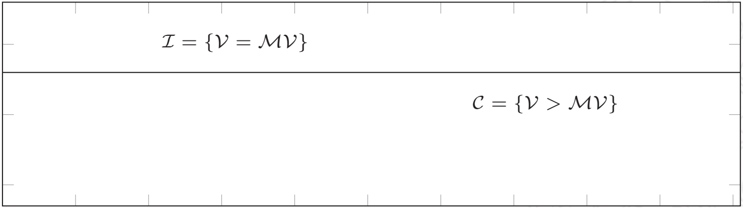
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Candidate Optimal Control

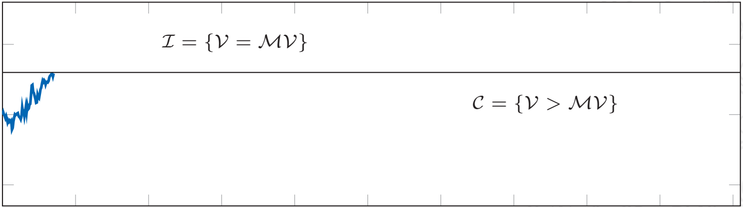
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Candidate Optimal Control

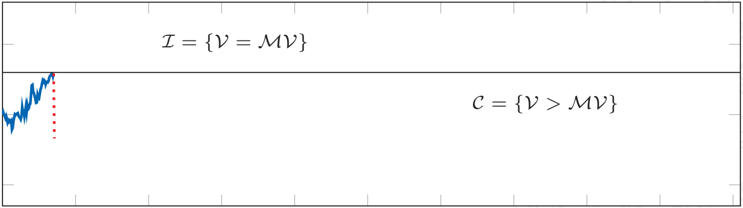
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Candidate Optimal Control

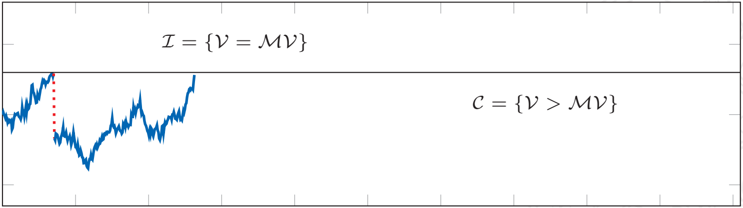
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Candidate Optimal Control

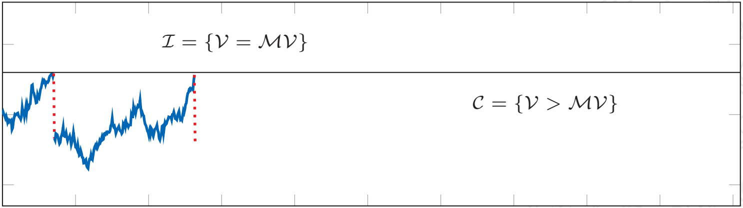
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Candidate Optimal Control

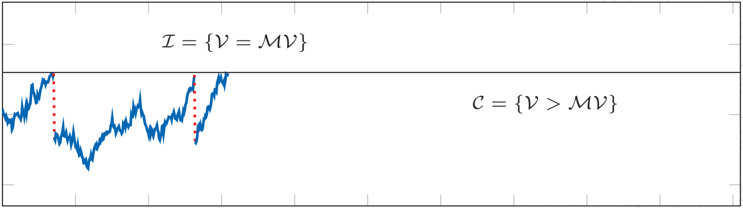
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Candidate Optimal Control

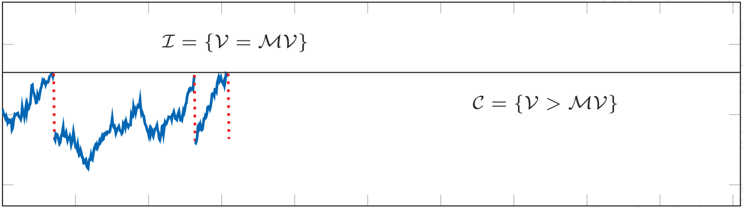
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Candidate Optimal Control

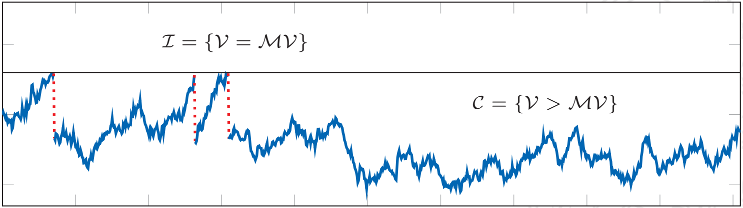
Define the so-called **maximum operator** $\mathcal{M}$ via

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward achieveable if you start with an impulse.

Observe that $\mathcal{V}(t,x) \geq \mathcal{M}\mathcal{V}(t,x)$ and

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
- the **optimal impulse** $\Delta^* \in Z(x)$ in state (t,x) should be chosen to be a maximizer for $\mathcal{M}\mathcal{V}(t,x)$.



A Formal Optimal Stopping Problem

By the **dynamic programming principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, X(\tau)\big)\big]$$

A Formal Optimal Stopping Problem

By the **dynamic programming principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, X(\tau)\big)\big] = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[G\big(\tau, X(\tau)\big)\big],$$

which is nothing but an **optimal stopping problem** with reward $G \triangleq \mathcal{M}\mathcal{V}$.

A Formal Optimal Stopping Problem

By the **dynamic programming principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, X(\tau)\big)\big] = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[G\big(\tau, X(\tau)\big)\big],$$

which is nothing but an **optimal stopping problem** with reward $G \triangleq \mathcal{M}\mathcal{V}$.

The general theory of optimal stopping lets us expect that

- $\mathcal{V}$ is equal to the **smallest superharmonic function** $\mathbb{V}$ dominating G, and

A Formal Optimal Stopping Problem

By the **dynamic programming principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, X(\tau)\big)\big] = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[G\big(\tau, X(\tau)\big)\big],$$

which is nothing but an **optimal stopping problem** with reward $G \triangleq \mathcal{M}\mathcal{V}$.

The general theory of optimal stopping lets us expect that

- $\mathcal{V}$ is equal to the **smallest superharmonic function** $\mathbb{V}$ dominating G, and
- if $\mathbb{V}$ is lower semi-continuous and G is upper semi-continuous, then $\mathbb{V} = \mathcal{V}$ and the first hitting time of the set $\{\mathbb{V} = G\} = \{\mathcal{V} = \mathcal{M}\mathcal{V}\}$ is **optimal**.

A Formal Optimal Stopping Problem

By the **dynamic programming principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, X(\tau)\big)\big] = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[G\big(\tau, X(\tau)\big)\big],$$

which is nothing but an **optimal stopping problem** with reward $G \triangleq \mathcal{M}\mathcal{V}$.

The general theory of optimal stopping lets us expect that

- $\mathcal{V}$ is equal to the **smallest superharmonic function** $\mathbb{V}$ dominating G, and
- if $\mathbb{V}$ is lower semi-continuous and G is upper semi-continuous, then $\mathbb{V} = \mathcal{V}$ and the first hitting time of the set $\{\mathbb{V} = G\} = \{\mathcal{V} = \mathcal{M}\mathcal{V}\}$ is **optimal**.

Remark: Under standard assumptions, $\mathcal{M}$ **preserves semi-continuity**, i.e. $G = \mathcal{M}\mathbb{V}$ is upper semi-continuous if $\mathbb{V}$ is upper semi-continuous. That is, continuity of $\mathbb{V}$ should suffice to solve the problem!

The Verification Theorem

Let $\mathbb{H}$ be the set of functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

The Verification Theorem

Let $\mathbb{H}$ be the set of functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process,

The Verification Theorem

Let $\mathbb{H}$ be the set of functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

The Verification Theorem

Let $\mathbb{H}$ be the set of functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

(H3) h satisfies the **terminal condition** $h(T, \cdot) \geq g$ on $\mathbb{R}^n$.

The Verification Theorem

Let $\mathbb{H}$ be the set of functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

(H3) h satisfies the **terminal condition** $h(T, \cdot) \geq g$ on $\mathbb{R}^n$.

Define $\mathbb{V} : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ to be the **pointwise infimum** of the members of $\mathbb{H}$.

The Verification Theorem

Let $\mathbb{H}$ be the set of functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

(H3) h satisfies the **terminal condition** $h(T, \cdot) \geq g$ on $\mathbb{R}^n$.

Define $\mathbb{V} : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ to be the **pointwise infimum** of the members of $\mathbb{H}$.

Verification "Theorem"

If $\mathbb{V}$ exists, is continuous, satisfies $\mathbb{V}(T, \cdot) = g$, and the candidate optimal control defined in terms of $\{\mathbb{V} = \mathcal{M}\mathbb{V}\}$ and $\{\mathbb{V} > \mathcal{M}\mathbb{V}\}$ is admissible, then (up to integrability) we have

$$\mathbb{V} = \mathcal{V} \qquad \text{and the candidate optimal control is optimal.}$$

The Verification Theorem

Let $\mathbb{H}$ be the set of functions $h : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

(H3) h satisfies the **terminal condition** $h(T,\cdot) \geq g$ on $\mathbb{R}^n$.

Define $\mathbb{V} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ to be the **pointwise infimum** of the members of $\mathbb{H}$.

Verification "Theorem"

If $\mathbb{V}$ exists, is continuous, satisfies $\mathbb{V}(T,\cdot) = g$, and the candidate optimal control defined in terms of $\{\mathbb{V} = \mathcal{M}\mathbb{V}\}$ and $\{\mathbb{V} > \mathcal{M}\mathbb{V}\}$ is admissible, then (up to integrability) we have

$$\mathbb{V} = \mathcal{V} \qquad \text{and the candidate optimal control is optimal.}$$

Proof: Relies mainly on classical optimal stopping techniques, i.e. works in quite general settings.

Continuity of the Value Function

How can we prove the continuity of $\mathbb{V}$?

Continuity of the Value Function

How can we prove the continuity of $\mathbb{V}$?

The smallest superharmonic function turns out to be a **viscosity solution** of the dynamic programming equation (DPE)

$$\min\{-\partial_t \mathbb{V}(t,x) - \mathcal{L}\mathbb{V}(t,x), \mathbb{V}(t,x) - \mathcal{M}\mathbb{V}(t,x)\} = 0 \quad \text{on } [0,T) \times \mathbb{R}^n,$$
$$\mathbb{V}(T,x) = g(x) \quad \text{on } \mathbb{R}^n,$$

where $\mathcal{L}$ denotes the **infinitesimal generator** of the uncontrolled state process X.

Continuity of the Value Function

How can we prove the continuity of $\mathbb{V}$?

The smallest superharmonic function turns out to be a **viscosity solution** of the dynamic programming equation (DPE)

$$\begin{aligned} \min\left\{-\partial_t \mathbb{V}(t,x) - \mathcal{L}\mathbb{V}(t,x), \mathbb{V}(t,x) - \mathcal{M}\mathbb{V}(t,x)\right\} &= 0 \qquad \text{on } [0,T) \times \mathbb{R}^n, \\ \mathbb{V}(T,x) &= g(x) \quad \text{on } \mathbb{R}^n, \end{aligned}$$

where $\mathcal{L}$ denotes the **infinitesimal generator** of the uncontrolled state process X.

More precisely,

- the **upper semi-continuous** envelope $\mathbb{V}^*$ is a viscosity **subsolution** and

Continuity of the Value Function

How can we prove the continuity of $\mathbb{V}$?

The smallest superharmonic function turns out to be a **viscosity solution** of the dynamic programming equation (DPE)

$$\min\left\{-\partial_t \mathbb{V}(t,x) - \mathcal{L}\mathbb{V}(t,x), \mathbb{V}(t,x) - \mathcal{M}\mathbb{V}(t,x)\right\} = 0 \qquad \text{on } [0,T) \times \mathbb{R}^n,$$
$$\mathbb{V}(T,x) = g(x) \quad \text{on } \mathbb{R}^n,$$

where $\mathcal{L}$ denotes the **infinitesimal generator** of the uncontrolled state process X.

More precisely,

- the **upper semi-continuous** envelope $\mathbb{V}^*$ is a viscosity **subsolution** and
- the **lower semi-continuous** envelope $\mathbb{V}_*$ is a viscosity **supersolution** and

Continuity of the Value Function

How can we prove the continuity of $\mathbb{V}$?

The smallest superharmonic function turns out to be a **viscosity solution** of the dynamic programming equation (DPE)

$$\begin{aligned} \min\left\{-\partial_t \mathbb{V}(t,x) - \mathcal{L}\mathbb{V}(t,x), \mathbb{V}(t,x) - \mathcal{M}\mathbb{V}(t,x)\right\} &= 0 \qquad \text{on } [0,T) \times \mathbb{R}^n, \\ \mathbb{V}(T,x) &= g(x) \quad \text{on } \mathbb{R}^n, \end{aligned}$$

where $\mathcal{L}$ denotes the **infinitesimal generator** of the uncontrolled state process X.

More precisely,

- the **upper semi-continuous** envelope $\mathbb{V}^*$ is a viscosity **subsolution** and
- the **lower semi-continuous** envelope $\mathbb{V}_*$ is a viscosity **supersolution** and

Continuity of $\mathbb{V}$ follows if the DPE admits a **comparison principle**.

The Comparison Argument

Comparison Principle

The DPE satisfies a **comparison principle** if the following holds: Let

The Comparison Argument

Comparison Principle

The DPE satisfies a **comparison principle** if the following holds: Let

(1) $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity supersolution of the DPE,

The Comparison Argument

Comparison Principle

The DPE satisfies a **comparison principle** if the following holds: Let

(1) $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u^*(T, \cdot) \leq v_*(T, \cdot)$ on $\mathbb{R}^n$,

The Comparison Argument

Comparison Principle

The DPE satisfies a **comparison principle** if the following holds: Let

(1) $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u^*(T,\cdot) \le v_*(T,\cdot)$ on $\mathbb{R}^n$,

(4) (possibly some growth conditions at infinity).

The Comparison Argument

Comparison Principle

The DPE satisfies a **comparison principle** if the following holds: Let

(1) $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u^*(T, \cdot) \leq v_*(T, \cdot)$ on $\mathbb{R}^n$,

(4) (possibly some growth conditions at infinity).

Then $u^* \leq v_*$ everywhere.

The Comparison Argument

Comparison Principle

The DPE satisfies a **comparison principle** if the following holds: Let

(1) $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u^*(T,\cdot) \leq v_*(T,\cdot)$ on $\mathbb{R}^n$,

(4) (possibly some growth conditions at infinity).

Then $u^* \leq v_*$ everywhere.

Thus, since $\mathbb{V}$ is a viscosity solution of the DPE, and if we have

$$\mathbb{V}^*(T,\cdot) \leq \mathbb{V}_*(T,\cdot) \qquad \text{on } \mathbb{R}^n,$$

then $\mathbb{V}^* \leq \mathbb{V}_*$ everywhere, i.e. $\mathbb{V}$ is **continuous**. The latter inequality can be verified if the terminal condition g is continuous.

The Comparison Argument

Comparison Principle

The DPE satisfies a **comparison principle** if the following holds: Let

(1) $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u^*(T, \cdot) \leq v_*(T, \cdot)$ on $\mathbb{R}^n$,

(4) (possibly some growth conditions at infinity).

Then $u^* \leq v_*$ everywhere.

Thus, since $\mathbb{V}$ is a viscosity solution of the DPE, and if we have

$$\mathbb{V}^*(T, \cdot) \leq \mathbb{V}_*(T, \cdot) \qquad \text{on } \mathbb{R}^n,$$

then $\mathbb{V}^* \leq \mathbb{V}_*$ everywhere, i.e. $\mathbb{V}$ is **continuous**. The latter inequality can be verified if the terminal condition g is continuous. The same argument furthermore implies **uniqueness** of $\mathbb{V}$.

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

(2) Show that $\mathbb{V}^*$ is a **viscosity subsolution** of the DPE. Can be done using the stochastic Perron's method.

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

(2) Show that $\mathbb{V}^*$ is a **viscosity subsolution** of the DPE. Can be done using the stochastic Perron's method.

(3) Show **continuity at time T**, i.e. $\mathbb{V}^*(T, \cdot) = \mathbb{V}_*(T, \cdot) = g$ on $\mathbb{R}^n$. Techniques for this are available and can be expected to work if g is continuous.

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

(2) Show that $\mathbb{V}^*$ is a **viscosity subsolution** of the DPE. Can be done using the stochastic Perron's method.

(3) Show **continuity at time T**, i.e. $\mathbb{V}^*(T,\cdot) = \mathbb{V}_*(T,\cdot) = g$ on $\mathbb{R}^n$. Techniques for this are available and can be expected to work if g is continuous.

(4) Verify that the DPE satisfies a **comparison principle**. Typically holds if there exists a strict classical supersolution of the DPE (candidates available).

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

(2) Show that $\mathbb{V}^*$ is a **viscosity subsolution** of the DPE. Can be done using the stochastic Perron's method.

(3) Show **continuity at time T**, i.e. $\mathbb{V}^*(T,\cdot) = \mathbb{V}_*(T,\cdot) = g$ on $\mathbb{R}^n$. Techniques for this are available and can be expected to work if g is continuous.

(4) Verify that the DPE satisfies a **comparison principle**. Typically holds if there exists a strict classical supersolution of the DPE (candidates available).

(5) It follows that $\mathbb{V}$ is **continuous**.

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

(2) Show that $\mathbb{V}^*$ is a **viscosity subsolution** of the DPE. Can be done using the stochastic Perron's method.

(3) Show **continuity at time T**, i.e. $\mathbb{V}^*(T,\cdot) = \mathbb{V}_*(T,\cdot) = g$ on $\mathbb{R}^n$. Techniques for this are available and can be expected to work if g is continuous.

(4) Verify that the DPE satisfies a **comparison principle**. Typically holds if there exists a strict classical supersolution of the DPE (candidates available).

(5) It follows that $\mathbb{V}$ is **continuous**.

(6) Define a candidate optimal control in terms of $\{\mathbb{V} = \mathcal{M}\mathbb{V}\}$ and $\{\mathbb{V} > \mathcal{M}\mathbb{V}\}$ and verify that the control is **admissible**. This is problem specific and can become quite difficult.

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

(2) Show that $\mathbb{V}^*$ is a **viscosity subsolution** of the DPE. Can be done using the stochastic Perron's method.

(3) Show **continuity at time T**, i.e. $\mathbb{V}^*(T,\cdot) = \mathbb{V}_*(T,\cdot) = g$ on $\mathbb{R}^n$. Techniques for this are available and can be expected to work if g is continuous.

(4) Verify that the DPE satisfies a **comparison principle**. Typically holds if there exists a strict classical supersolution of the DPE (candidates available).

(5) It follows that $\mathbb{V}$ is **continuous**.

(6) Define a candidate optimal control in terms of $\{\mathbb{V} = \mathcal{M}\mathbb{V}\}$ and $\{\mathbb{V} > \mathcal{M}\mathbb{V}\}$ and verify that the control is **admissible**. This is problem specific and can become quite difficult.

(7) Apply the **verification theorem** to conclude that $\mathbb{V} = \mathcal{V}$ and obtain the optimality of the control.

Summing Up the Procedure

Summing up the ideas, impulse control problems can be solved as follows:

(1) Show that $\mathbb{V}_*$ is a **viscosity supersolution** of the DPE. This is typically easy and can be done by classical viscosity arguments.

(2) Show that $\mathbb{V}^*$ is a **viscosity subsolution** of the DPE. Can be done using the stochastic Perron's method.

(3) Show **continuity at time T**, i.e. $\mathbb{V}^*(T,\cdot) = \mathbb{V}_*(T,\cdot) = g$ on $\mathbb{R}^n$. Techniques for this are available and can be expected to work if g is continuous.

(4) Verify that the DPE satisfies a **comparison principle**. Typically holds if there exists a strict classical supersolution of the DPE (candidates available).

(5) It follows that $\mathbb{V}$ is **continuous**.

(6) Define a candidate optimal control in terms of $\{\mathbb{V} = \mathcal{M}\mathbb{V}\}$ and $\{\mathbb{V} > \mathcal{M}\mathbb{V}\}$ and verify that the control is **admissible**. This is problem specific and can become quite difficult.

(7) Apply the **verification theorem** to conclude that $\mathbb{V} = \mathcal{V}$ and obtain the optimality of the control.

(8) Be happy! You just solved the problem.

Utility Maximization with Constant Costs

Based on:
Belak, C. and Christensen, S.: **Utility maximization in a factor model with constant and proportional costs**, Preprint, available on SSRN, 2017.

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$
\begin{aligned}
\mathrm{d}X_1(t) &= rX_1(t)\mathrm{d}t, && t \in [\tau^k, \tau^{k+1}), \\
\mathrm{d}X_2(t) &= \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), && t \in [\tau^k, \tau^{k+1}),
\end{aligned}
$$

Remark: The model can be generalized (more than one risky asset, factor processes).

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned}
\mathrm{d}X_1(t) &= rX_1(t)\mathrm{d}t, && t \in [\tau^k, \tau^{k+1}), \\
\mathrm{d}X_2(t) &= \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), && t \in [\tau^k, \tau^{k+1}), \\
X_1(\tau_k) &= X_1(\tau_k-) - \Delta_k - \gamma|\Delta_k| - C, \\
X_2(\tau_k) &= X_2(\tau_k-) + \Delta_k,
\end{aligned}$$

where $\gamma \in (0,1)$ (**proportional cost**) and $C > 0$ (**constant cost**).

Remark: The model can be generalized (more than one risky asset, factor processes).

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned}
\mathrm{d}X_1(t) &= rX_1(t)\mathrm{d}t, && t \in [\tau^k, \tau^{k+1}),\\
\mathrm{d}X_2(t) &= \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), && t \in [\tau^k, \tau^{k+1}),\\
X_1(\tau_k) &= X_1(\tau_k-) - \Delta_k - \gamma|\Delta_k| - C,\\
X_2(\tau_k) &= X_2(\tau_k-) + \Delta_k,
\end{aligned}$$

where $\gamma \in (0,1)$ (**proportional cost**) and $C > 0$ (**constant cost**).

A portfolio $x \in \mathbb{R}^n$ is **solvent** if it has a **positive liquidation** value $\mathrm{L}(x)$, i.e.,

$$\mathrm{L}(x) \triangleq x_1 + x_2 - \gamma|x_2| - C\mathbb{1}_{\{x_2\neq 0\}} > 0.$$

The set $\mathcal{S} \subset \mathbb{R}^2$ of solvent portfolios is called the **solvency region**.

Remark: The model can be generalized (more than one risky asset, factor processes).

The Optimization Criterion

Now fix a **utility function** $U : \mathbb{R}_+ \to \mathbb{R}$ such that

- U is strictly increasing, continuous, and concave,
- U is lower bounded; without loss of generality $U(0) = 0$,
- U satisfies $U(l) \leq M(1 + |l|^p)$ for some $M > 0$, $p \in (0, 1)$.

The Optimization Criterion

Now fix a **utility function** $U : \mathbb{R}_+ \to \mathbb{R}$ such that

- U is strictly increasing, continuous, and concave,
- U is lower bounded; without loss of generality $U(0) = 0$,
- U satisfies $U(l) \leq M(1 + |l|^p)$ for some $M > 0$, $p \in (0, 1)$.

The objective is to **maximize utility of terminal wealth**, i.e.

$$\mathcal{V}(t,x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[U\Big(\mathrm{L}\big(X_{t,x}^{\Lambda}(\tau_{\mathcal{S}}^{\Lambda} \wedge T)\big)\Big)\Big],$$

where

- $\mathcal{A}(t,x)$ denotes the set of **admissible strategies** Λ for the initial state (t,x).
- $\tau_{\mathcal{S}}^{\Lambda}$ denotes the **bankruptcy time** corresponding to the strategy Λ, i.e. the first exit time of $X_{t,x}^{\Lambda}$ from the solvency region $\mathcal{S}$.

Additional Difficulties in this Model

The general results cannot be applied directly since

- … the **terminal condition is discontinuous** since $\mathcal{V}(T,x) = U(\mathrm{L}(x))$ and

$$\mathrm{L}(x) \triangleq x_1 + x_2 - \gamma|x_2| - C\mathbb{1}_{\{x_2 \neq 0\}}.$$

  is discontinuous.

Additional Difficulties in this Model

The general results cannot be applied directly since

- ... the **terminal condition is discontinuous** since $\mathcal{V}(T, x) = U(\mathrm{L}(x))$ and

$$\mathrm{L}(x) \triangleq x_1 + x_2 - \gamma|x_2| - C\mathbb{1}_{\{x_2 \neq 0\}}.$$

  is discontinuous.
- ... the **state space is constrained** to $\mathcal{S}$ instead of all of $\mathbb{R}^2$, meaning we face additional boundary conditions on $\partial\mathcal{S}$.

Additional Difficulties in this Model

The general results cannot be applied directly since

- ... the **terminal condition is discontinuous** since $\mathcal{V}(T, x) = U(\mathrm{L}(x))$ and

  $$\mathrm{L}(x) \triangleq x_1 + x_2 - \gamma|x_2| - C\mathbb{1}_{\{x_2 \neq 0\}}.$$

  is discontinuous.
- ... the **state space is constrained** to $\mathcal{S}$ instead of all of $\mathbb{R}^2$, meaning we face additional boundary conditions on $\partial\mathcal{S}$.
- ... the set of transactions may be **empty**, the maximum operator $\mathcal{M}$ does not preserve **semi-continuity** everywhere, ...

Additional Difficulties in this Model

The general results cannot be applied directly since

- ... the **terminal condition is discontinuous** since $\mathcal{V}(T, x) = U(\mathrm{L}(x))$ and

  $$\mathrm{L}(x) \triangleq x_1 + x_2 - \gamma |x_2| - C \mathbb{1}_{\{x_2 \neq 0\}}.$$

  is discontinuous.
- ... the **state space is constrained** to $\mathcal{S}$ instead of all of $\mathbb{R}^2$, meaning we face additional boundary conditions on $\partial \mathcal{S}$.
- ... the set of transactions may be **empty**, the maximum operator $\mathcal{M}$ does not preserve **semi-continuity** everywhere, ...

Except for the first issue, these problems can be dealt with. The discontinuity at time T is however a deal breaker.

Additional Difficulties in this Model

The general results cannot be applied directly since

- ... the **terminal condition is discontinuous** since $\mathcal{V}(T,x) = U(\mathrm{L}(x))$ and

  $$\mathrm{L}(x) \triangleq x_1 + x_2 - \gamma|x_2| - C\mathbb{1}_{\{x_2 \neq 0\}}.$$

  is discontinuous.
- ... the **state space is constrained** to $\mathcal{S}$ instead of all of $\mathbb{R}^2$, meaning we face additional boundary conditions on $\partial\mathcal{S}$.
- ... the set of transactions may be **empty**, the maximum operator $\mathcal{M}$ does not preserve **semi-continuity** everywhere, ...

Except for the first issue, these problems can be dealt with. The discontinuity at time T is however a deal breaker. Our **resolution**: Work with the following liquidation function instead:

$$\mathrm{L}(x) \triangleq x_1 + x_2 - \gamma|x_2| - C.$$

In this case the general solution strategy can be adapted to solve the problem...

Or so we thought...

(Dis-)Continuity of the Value Function

The Discontinuity on the Boundary

Even with the adjusted liquidation function, the value function has a **discontinuity on the boundary $\partial\mathcal{S}$** of the state space where $x_2 = 0$. Hence the arguments no longer work.

The Discontinuity on the Boundary

Even with the adjusted liquidation function, the value function has a **discontinuity on the boundary $\partial\mathcal{S}$** of the state space where $x_2 = 0$. Hence the arguments no longer work. More precisely:

- The comparison principle does not apply to $\mathbb{V}$.

The Discontinuity on the Boundary

Even with the adjusted liquidation function, the value function has a **discontinuity on the boundary $\partial\mathcal{S}$** of the state space where $x_2 = 0$. Hence the arguments no longer work. More precisely:

- The comparison principle does not apply to $\mathbb{V}$.
- The verification theorem cannot be applied.

The Discontinuity on the Boundary

Even with the adjusted liquidation function, the value function has a **discontinuity on the boundary $\partial\mathcal{S}$** of the state space where $x_2 = 0$. Hence the arguments no longer work. More precisely:

- The comparison principle does not apply to $\mathbb{V}$.
- The verification theorem cannot be applied.

The hot fix: Replace the bond with a stock, i.e. add a diffusion component to X_1. This gives continuity on the boundary. But can we do without this assumption?

The Discontinuity on the Boundary

Even with the adjusted liquidation function, the value function has a **discontinuity on the boundary $\partial\mathcal{S}$** of the state space where $x_2 = 0$. Hence the arguments no longer work. More precisely:

- The comparison principle does not apply to $\mathbb{V}$.
- The verification theorem cannot be applied.

The hot fix: Replace the bond with a stock, i.e. add a diffusion component to X_1. This gives continuity on the boundary. But can we do without this assumption?

Observation: The discontinuity occurs on the x_1 axis, and the uncontrolled state process is never able to cross this axis.

The Discontinuity on the Boundary

Even with the adjusted liquidation function, the value function has a **discontinuity on the boundary $\partial\mathcal{S}$** of the state space where $x_2 = 0$. Hence the arguments no longer work. More precisely:

- The comparison principle does not apply to $\mathbb{V}$.
- The verification theorem cannot be applied.

The hot fix: Replace the bond with a stock, i.e. add a diffusion component to X_1. This gives continuity on the boundary. But can we do without this assumption?

Observation: The discontinuity occurs on the x_1 axis, and the uncontrolled state process is never able to cross this axis.

Idea: If we can prove continuity of $\mathbb{V}$ on each quadrant and on each axis separately, then the verification theorem still works. But how to get this **piecewise continuity**?

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

The Local Comparison Principle

For a locally bounded function $u : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0, T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity supersolution of the DPE,

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u_i^*(T,\cdot) \le v_*^i(T,\cdot)$ on $\mathcal{S}_i$, $i = 1, \ldots, 5$,

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u_i^*(T,\cdot) \leq v_*^i(T,\cdot)$ on $\mathcal{S}_i$, $i = 1, \ldots, 5$,

(4) $u_i^* \leq v_*^i$ on $[0,T] \times (\partial\mathcal{S}_i \cap \partial\mathcal{S})$, $i = 1, \ldots, 5$,

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u_i^*(T,\cdot) \leq v_*^i(T,\cdot)$ on $\mathcal{S}_i$, $i = 1, \ldots, 5$,

(4) $u_i^* \leq v_*^i$ on $[0,T] \times (\partial\mathcal{S}_i \cap \partial\mathcal{S})$, $i = 1, \ldots, 5$,

(5) (growth conditions).

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \dots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u_i^*(T, \cdot) \leq v_*^i(T, \cdot)$ on $\mathcal{S}_i$, $i = 1, \dots, 5$,

(4) $u_i^* \leq v_*^i$ on $[0,T] \times (\partial\mathcal{S}_i \cap \partial\mathcal{S})$, $i = 1, \dots, 5$,

(5) (growth conditions).

Then $u_i^* \leq v_*^i$ on $[0,T] \times \mathcal{S}_i$, $i = 1, \dots, 5$.

The Local Comparison Principle

For a locally bounded function $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u_i^*(T,\cdot) \leq v_*^i(T,\cdot)$ on $\mathcal{S}_i$, $i = 1, \ldots, 5$,

(4) $u_i^* \leq v_*^i$ on $[0,T] \times (\partial\mathcal{S}_i \cap \partial\mathcal{S})$, $i = 1, \ldots, 5$,

(5) (growth conditions).

Then $u_i^* \leq v_*^i$ on $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

This implies that any viscosity solution with piecewise continuous boundary condition is continuous if restricted to $[0,T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$. The viscosity solution need not be globally continuous.

The Local Comparison Principle

For a locally bounded function $u : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$, denote by u_i^* and u_*^i the upper and lower semi-continuous envelopes of u **restricted to** $[0, T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

It turn out that the DPE satisfies the following **local comparison principle**: Let

(1) $u : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity subsolution of the DPE,

(2) $v : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a viscosity supersolution of the DPE,

(3) $u_i^*(T, \cdot) \leq v_*^i(T, \cdot)$ on $\mathcal{S}_i$, $i = 1, \ldots, 5$,

(4) $u_i^* \leq v_*^i$ on $[0, T] \times (\partial\mathcal{S}_i \cap \partial\mathcal{S})$, $i = 1, \ldots, 5$,

(5) (growth conditions).

Then $u_i^* \leq v_*^i$ on $[0, T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$.

This implies that any viscosity solution with piecewise continuous boundary condition is continuous if restricted to $[0, T] \times \mathcal{S}_i$, $i = 1, \ldots, 5$. The viscosity solution need not be globally continuous.

Still open: Showing that $\mathbb{V}$ satisfies **(4)**. Take it as an assumption for now.

Conclusion

- **Aim**: Solve a portfolio optimization problem involving constant costs.

Conclusion

- **Aim**: Solve a portfolio optimization problem involving constant costs.
- We devise a new verification procedure based on **superharmonic functions** and **viscosity solutions** to solve general impulse control problems, provided that the value function is continuous.

Conclusion

- **Aim**: Solve a portfolio optimization problem involving constant costs.
- We devise a new verification procedure based on **superharmonic functions** and **viscosity solutions** to solve general impulse control problems, provided that the value function is continuous.
- Since the value function of the portfolio optimization problem is not continuous, we use a **localization technique** to extend the procedure to **piecewise continuous** functions.

Conclusion

- **Aim**: Solve a portfolio optimization problem involving constant costs.
- We devise a new verification procedure based on **superharmonic functions** and **viscosity solutions** to solve general impulse control problems, provided that the value function is continuous.
- Since the value function of the portfolio optimization problem is not continuous, we use a **localization technique** to extend the procedure to **piecewise continuous** functions.
- In particular, we manage to solve a control problem with **discontinuous value function** and we obtain **uniqueness** of viscosity solutions with a **discontinuous boundary condition**.

Conclusion

- **Aim**: Solve a portfolio optimization problem involving constant costs.
- We devise a new verification procedure based on **superharmonic functions** and **viscosity solutions** to solve general impulse control problems, provided that the value function is continuous.
- Since the value function of the portfolio optimization problem is not continuous, we use a **localization technique** to extend the procedure to **piecewise continuous** functions.
- In particular, we manage to solve a control problem with **discontinuous value function** and we obtain **uniqueness** of viscosity solutions with a **discontinuous boundary condition**.

Thanks for your attention!

Optimal Trading Regions