Universität Trier

Utility Maximization with Constant Costs

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Joint work with **Sören Christensen** (University of Hamburg).

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Market Model and Problem Formulation

The Market Model and Transaction Costs

The **financial market** consists of $n+1$ assets $P = (P^0, P^1, \dots, P^n)$ with prices

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where $\gamma_i \in (0,1)$ and $K_i > 0$ for $i = 1, \dots, n$.

Solvency and Trading Strategies

A portfolio $x \in \mathbb{R}^{n+1}$ is **solvent** if it has a positive **liquidation value** $L(x)$, i.e.

$$L(x) \triangleq x_0 + \sum_{i=1}^{n} [x_i - \gamma_i |x_i| - K_i] > 0.$$

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In the above,

- τ_{Λ} denotes the **bankruptcy time** of the strategy Λ, i.e. the first time X^{Λ} leaves the solvency region $\mathcal{S}$, and
- $\mathcal{A}(t, x, y)$ denotes the set of admissible strategies corresponding to an initial state of (t, x, y).

Related Literature

In the existing literature, there are three different approaches:

- **Classical Verification**: Eastham and Hastings (1988), Korn (1998), Bielecki and Pliska (2000), Liu (2004).
- **Viscosity Characterization**: Oksendal and Sulem (2002), Altarovici, Reppen, and Soner (2016)
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Note: To the best of our knowledge, not even in the Black-Scholes case with log or power utility, a rigorous existence result for optimal strategies is known.

The Candidate Optimal Strategy

The Quasi-Variational Inequalities

The Bellman principle suggests that $\mathcal{V}$ can be linked to the **quasi-variational inequalities** (QVIs)

$$\min\Big\{-\partial_t \mathcal{V}(t,x,y) - \mathcal{L}\mathcal{V}(t,x,y), \mathcal{V}(t,x,y) - \mathcal{M}\mathcal{V}(t,x,y)\Big\} = 0$$

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Note: $\mathcal{M}\mathcal{V}(t,x,y)$ can be thought of as the optimal expected utility if the investor is **forced to make a transaction** at time t.

A Candidate Optimal Control

Observe that

- if $\mathcal{V}(t,x,y) > \mathcal{MV}(t,x,y)$, a transaction **cannot be optimal**,
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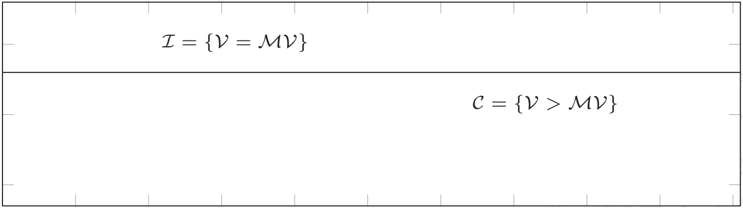

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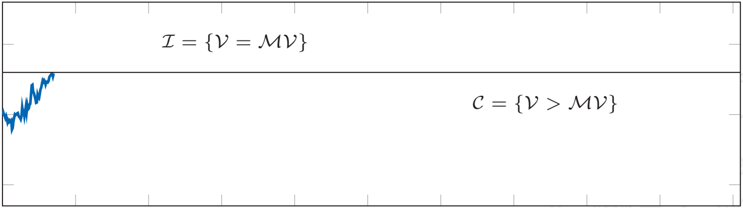

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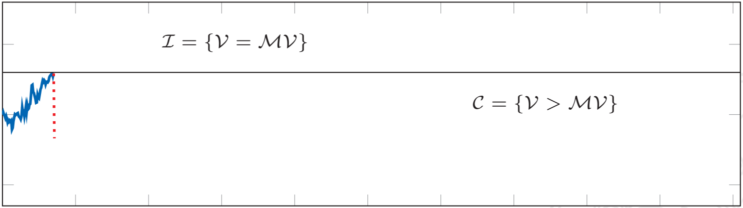

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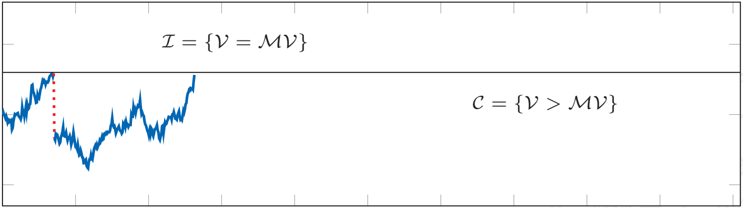

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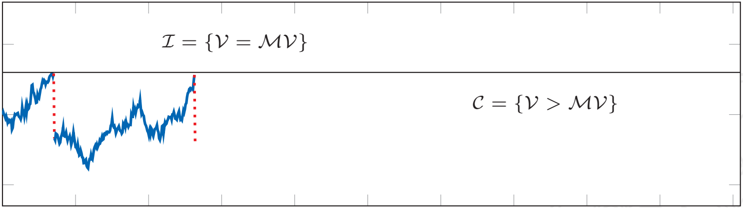

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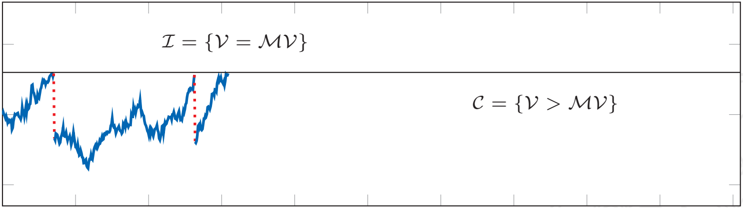

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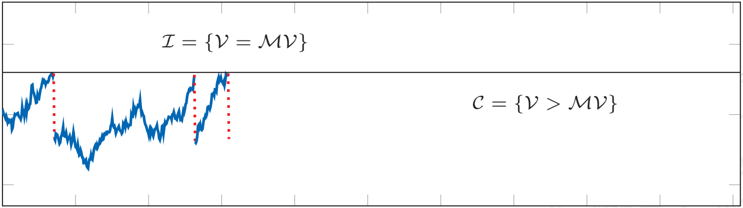

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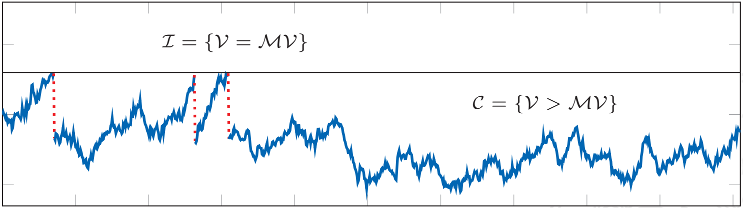

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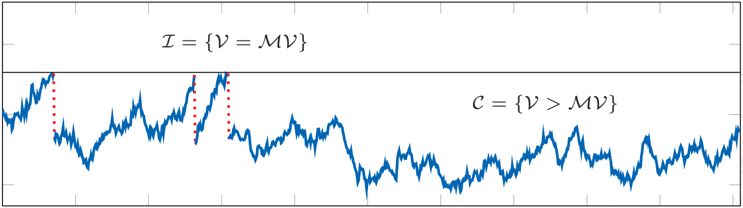

Problem: To verify optimality we need sufficient regularity to apply Itô's formula, but it seems unlikely that the QVIs admit a $C^{1,2}$-solution.

Verification via Superharmonic Functions

A Formal Optimal Stopping Problem

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Remark: $G = \mathcal{M}\mathcal{V}$ is upper semi-continuous if $\mathcal{V} = \mathbb{V}$ is upper semi-continuous. That is, to verify optimality, continuity of $\mathbb{V}$ should suffice!

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Moreover, define the **pointwise infimum** $\mathbb{V} = \inf_{h \in \mathbb{H}} h$.

The Set of Superharmonic Functions

Define $\mathbb{H}$ to be the set of all $h : [0, T] \times \overline{\mathcal{S}} \times \mathbb{R}^d \to [0, \infty)$ such that

- h is **superharmonic** with respect to the uncontrolled state process,
- h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,
- $h \geq \mathcal{V}$ on the parabolic boundary, and
- h is **continuous**.

Moreover, define the **pointwise infimum** $\mathbb{V} = \inf_{h \in \mathbb{H}} h$.

Verification Theorem (Belak and Christensen (2016))

Suppose that $\mathbb{V}$ is continuous. Then $\mathbb{V} = \mathcal{V}$ and the candidate optimal control Λ^* defined in terms of the sets

$$\mathcal{C} = \{\mathbb{V} > \mathcal{M}\mathbb{V}\} \quad \text{and} \quad \mathcal{I} = \{\mathbb{V} = \mathcal{M}\mathbb{V}\}.$$

is admissible and optimal.

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(4) By **viscosity comparison**: $\mathbb{V} \leq \mathbb{V}_*$, i.e. $\mathbb{V}$ is continuous and the unique viscosity solution of the QVIs.

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- It allows us to verify optimality of controls **without any regularity** of the value function beyond **continuity**.
- The approach **generalizes** to a big class of general impulse control problems (see Belak, Christensen, and Seifried (2017)).

Thanks for your attention!

Belak and Christensen (2016):
Utility Maximization in a Factor Model with Constant and Proportional Costs

Belak, Christensen, and Seifried (2017):
A General Verification Result for Stochastic Impulse Control Problems
SIAM J. Control Optim., Vol. 55, No. 2, pp. 627–649

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