Non-Smooth Verification for Impulse Control Problems

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Joint work with **Sören Christensen** (Hamburg) and **Frank Seifried** (Trier).

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Outline

(1) Motivation: Portfolio Optimization with Proportional and Constant Costs

(2) The General Impulse Control Problem

(3) Non-Smooth Verification via Superharmonic Functions

(4) Application to the Transaction Cost Problem

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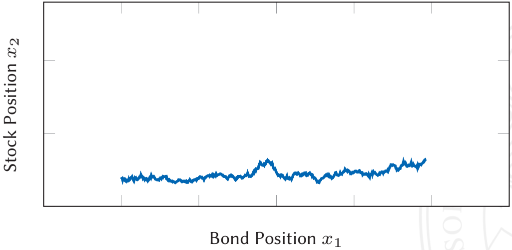

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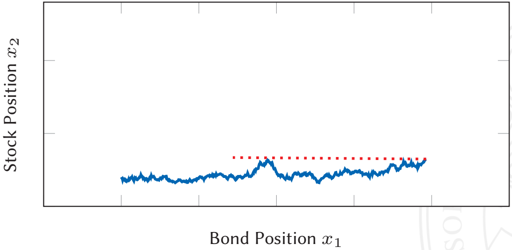

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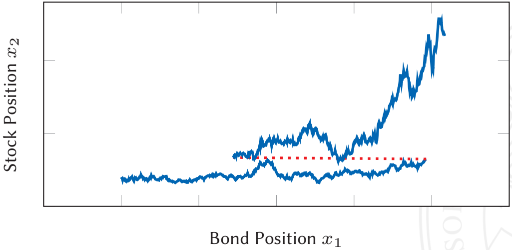

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- **Classical verification**: Eastham and Hastings (1988), Korn (1998), Bielecki and Pliska (2000), Liu (2004).
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To the best of our knowledge, existence of optimal controls is **unknown** — even in a Black-Scholes setting with $\log$- or power utility.

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\mathrm{d}X(t) &= \mu(X(t))\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \qquad t \in [\tau_k, \tau_{k+1}), \\
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- the stopping times τ_k are **increasing** and **do not accumulate** in that

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The **objective** is to find a maximizer of

$$\mathcal{V}(t,x) = \sup_{\Lambda\in\mathcal{A}(t,x)} \mathbb{E}\Big[\sum_{k\in\mathbb{N}} K(X^\Lambda_{t,x}(\tau_k-), \Delta_k)\mathbb{1}_{\{\tau_k\leq T\}} + g(X^\Lambda_{t,x}(T))\Big].$$

The Quasi-Variational Inequalities

The martingale optimality principle of stochastic control suggests that the value function $\mathcal{V}$ can be linked to the **quasi-variational inequalities** (QVIs)

$$\begin{aligned} \min\{-\partial_t\mathcal{V} - \mathcal{L}\mathcal{V}, \mathcal{V} - \mathcal{M}\mathcal{V}\} &= 0 && \text{on } [0,T)\times\mathbb{R}^n, \\ \mathcal{V}(T,\cdot) &= g && \text{on } \mathbb{R}^n, \end{aligned}$$

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- and the operator $\mathcal{M}$ is the **maximum operator** given by

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \big[\mathcal{V}\big(t, \Gamma(x,\Delta)\big) + K(x,\Delta)\big].$$

A Candidate Optimal Control

Observe that

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
- if $\mathcal{V}(t,x) = \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) is expected to be **optimal**.
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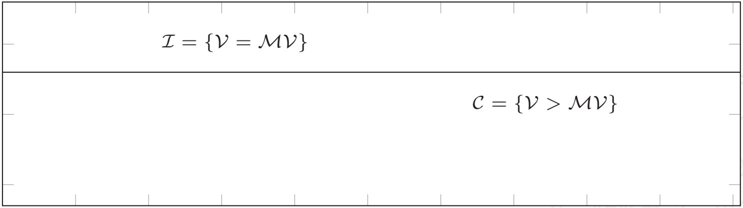

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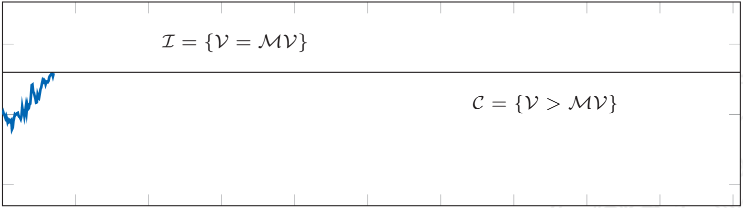

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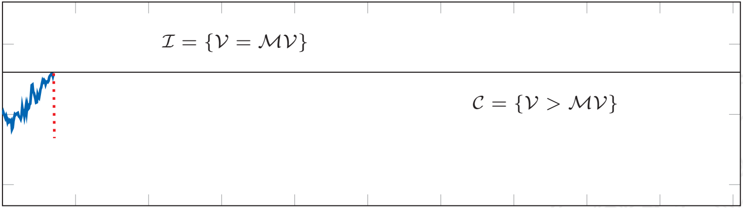

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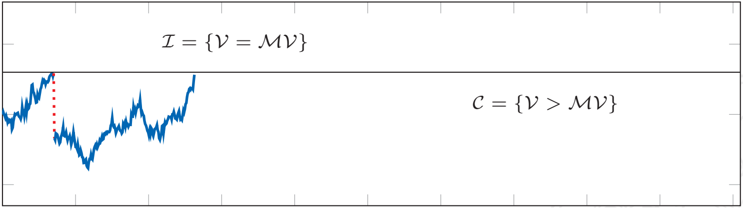

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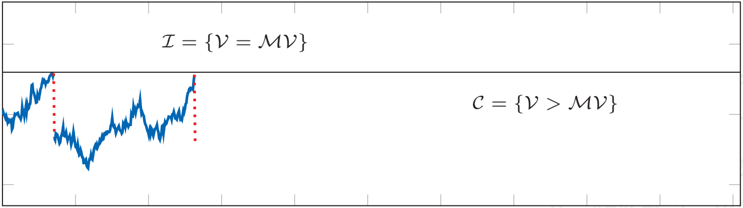

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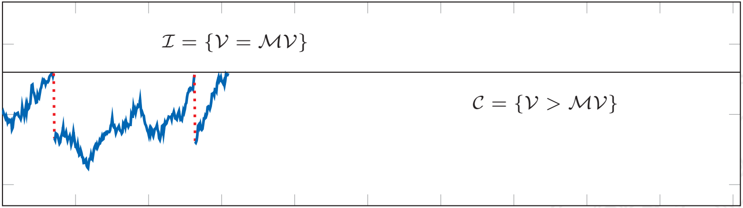

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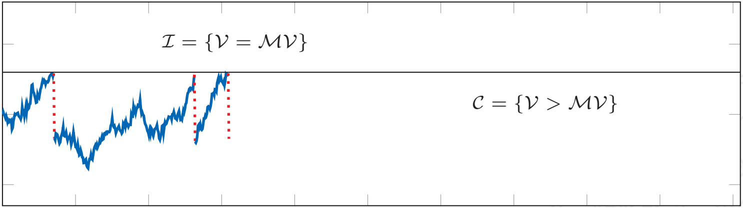

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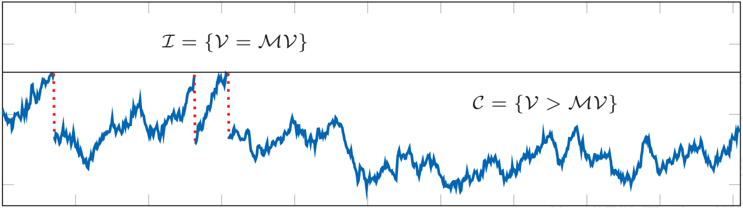

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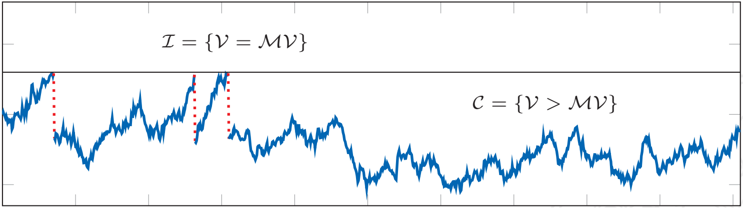

Problem: Verification requires a solution of the QVIs which is sufficiently smooth to apply Itô's formula.

Non-Smooth Verification via Superharmonic Functions

A Formal Optimal Stopping Problem

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Remark: Under standard assumptions, $G = \mathcal{M}\mathbb{V}$ is upper semi-continuous if $\mathbb{V}$ is upper semi-continuous. That is, to verify optimality, continuity of $\mathbb{V}$ should suffice!

The Verification Theorem

Let $\mathbb{H}$ be the set of **upper semi-continuous** functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

- **(H1)** h is **superharmonic** with respect to the uncontrolled state process,
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Note: If $\mathcal{V}$ is known to be continuous, then $\mathcal{V} \in \mathbb{H}$ and things become easy.

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(4) Then $\mathfrak{V} \le \mathcal{V} \le \mathbb{V}$. Now apply viscosity comparison so that $\mathfrak{V} \ge \mathbb{V}$ and hence

$$\mathcal{V} = \mathbb{V} = \mathfrak{V}$$

is **continuous** and the **unique viscosity solution** of the QVIs.

Sufficient Conditions

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- μ, σ, Γ Lipschitz in the state variable; Γ, Z, K, g continuous
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Note: The existence of Ψ and viscosity comparison is closely related to the existence of a strict supersolution of the QVIs. Candidates for Ψ are available.

Further Remarks on our Approach

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- If **continuity** of the value function is known a priori, the stochastic Perron method can be skipped.
- Proving **admissibility** can be difficult and has to be done case-by-case.
- **Advantage** over the classical approach: The candidate optimal control is defined in terms of the value function, making it easier to verify admissibility (at least in principle).

Solving the Transaction Cost Problem

The Market Model

We assume that the portfolio $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned}
\mathrm{d}Y(t) &= \alpha(Y(t))\mathrm{d}t + \beta(Y(t))\mathrm{d}W(t), && t \in [0,T],\\
\mathrm{d}X(t) &= \mathrm{diag}(X(t))\big[\mu(Y(t))\mathrm{d}t + \sigma(Y(t))\mathrm{d}W(t)\big], && t \in [\tau^k, \tau^{k+1}),\\
X(\tau^k) &= \Gamma(X(\tau^k-), \Delta^k),
\end{aligned}$$

where $\Gamma(x,\Delta) = \xi$ with

$$\begin{aligned}
\xi_1 &= x_1 - \textstyle\sum_{j=2}^n [\Delta_j + \gamma_j|\Delta_j| + K_j]\mathbb{1}_{\{\Delta_j \neq 0\}},\\
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A portfolio $x \in \mathbb{R}^n$ is **solvent** if it has a **positive liquidation** value $\mathrm{L}(x)$, i.e.,

$$\mathrm{L}(x) \triangleq x_1 + \textstyle\sum_{j=2}^{n}[x_j - \gamma_j|x_j| - K_j] > 0.$$

The set $\mathcal{S} \subset \mathbb{R}^n$ of solvent portfolios is called the **solvency region**.

The Optimization Criterion

Now fix a **utility function** $U : \mathbb{R}_+ \to \mathbb{R}$ such that

- U is strictly increasing and continuous (not necessarily concave),
- U is lower bounded; without loss of generality $U(0) = 0$,
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The objective is to **maximize utility of terminal wealth**, i.e.

$$\mathcal{V}(t,x,y) = \sup_{\Lambda \in \mathcal{A}(t,x,y)} \mathbb{E}\Big[U\Big(\mathrm{L}\big(X^{\Lambda}_{t,x,y}(\tau^{\Lambda}_{\mathcal{S}} \wedge T)\big)\Big)\Big],$$

where

- $\mathcal{A}(t,x,y)$ denotes the set of **admissible strategies** Λ corresponding to the initial state (t,x,y).
- $\tau^{\Lambda}_{\mathcal{S}}$ denotes the **bankruptcy time** corresponding to the strategy Λ, i.e. the first exit time of $X^{\Lambda}_{t,x,y}$ from the solvency region $\mathcal{S}$.

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- Viscosity comparison and non-emptiness of $\mathbb{H}$ hold since the **frictionless value function** is a classical supersolution of the QVIs.
- Admissibility of the candidate optimal strategy can be verified. **Idea**: The investor goes bankrupt before the trading times accumulate.

Conclusion and Outlook

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- We devise a new verification procedure based on **superharmonic functions**, **viscosity solutions**, and the **stochastic Perron method**.
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- The results open the door to a lot of interesting questions...

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Thanks for your attention!