Universität Trier

Utility Maximization with Constant Costs

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Joint work with **Sören Christensen** (University of Hamburg) and **Frank Seifried** (University of Trier).

11th Bachelier Colloquium, Métabief, France
January 16, 2017

The General Impulse Control Problem

Consider an $\mathbb{R}^n$-valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} \mathrm{d}X(t) &= \mu(X(t))\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \qquad t \in [\tau_k, \tau_{k+1}), \\ X(\tau_k) &= \Gamma(X(\tau_k-), \Delta_k), \end{aligned}$$

where

- the stopping times τ_k **do not accumulate**, i.e. $\mathbb{P}[\lim_{k\to\infty} \tau_k > T] = 1$,
- the impulses Δ_k are chosen from a set $Z(X(\tau_k-)) \subset \mathbb{R}^m$.

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The **objective** is to maximize

$$\mathcal{V}(t,x) = \sup_{\Lambda\in\mathcal{A}(t,x)} \mathbb{E}\Big[\sum_{k\in\mathbb{N}} K(X^\Lambda_{t,x}(\tau_k-), \Delta_k)\mathbb{1}_{\{\tau_k\le T\}} + g(X^\Lambda_{t,x}(T))\Big].$$

The Quasi-Variational Inequalities

The martingale optimality principle of stochastic control suggests that the value function $\mathcal{V}$ can be linked to the **quasi-variational inequalities** (QVIs)

$$\begin{aligned} \min\{-\partial_t \mathcal{V} - \mathcal{L}\mathcal{V}, \mathcal{V} - \mathcal{M}\mathcal{V}\} &= 0 && \text{on } [0,T) \times \mathbb{R}^n, \\ \mathcal{V}(T,\cdot) &= g && \text{on } \mathbb{R}^n, \end{aligned}$$

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- the operator $\mathcal{L}$ is the **infinitesimal generator** given by

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- and the operator $\mathcal{M}$ is the **maximum operator** given by

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \big[\mathcal{V}\big(t, \Gamma(x,\Delta)\big) + K(x,\Delta)\big].$$

A Candidate Optimal Control

Observe that

- if $\mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)$, an impulse in state (t,x) **cannot be optimal**, and
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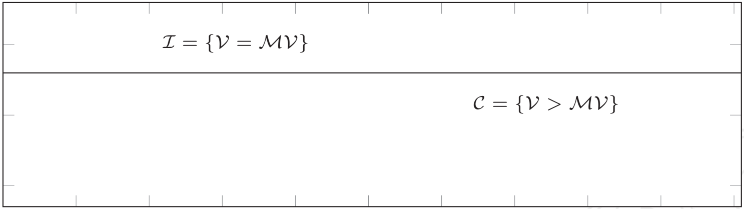

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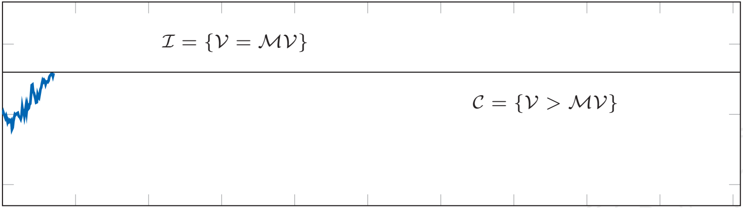

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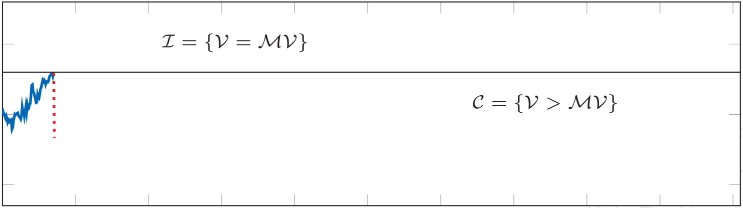

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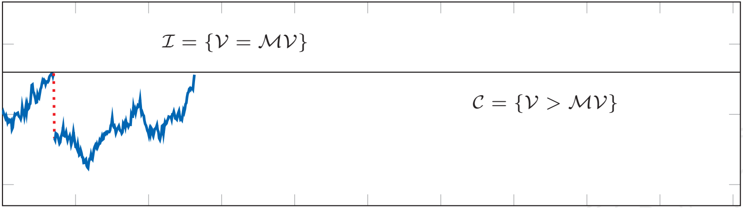

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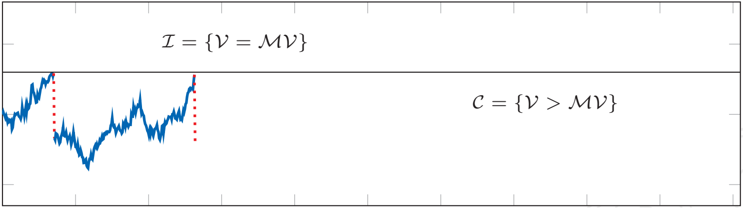

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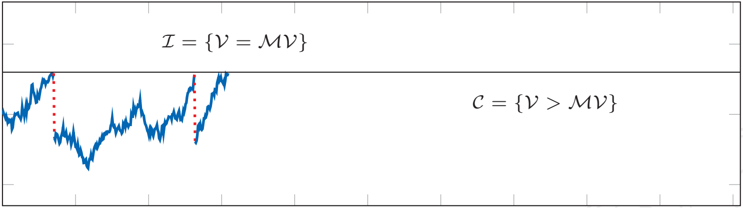

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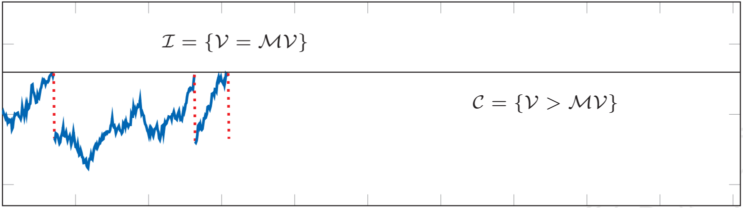

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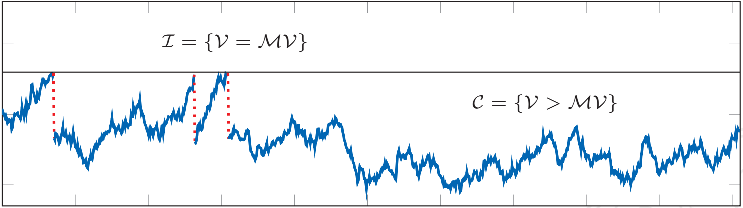

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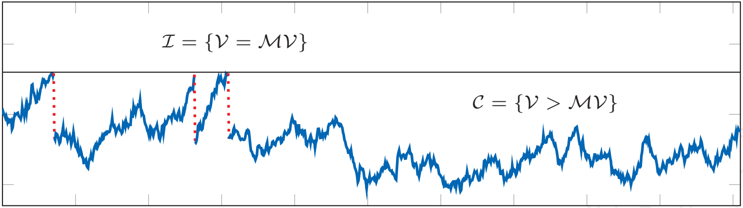

Problem: Verification requires a $C^{1,2}$-solution of the QVIs.

A Formal Optimal Stopping Problem

By the **dynamic programming principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, X_{t,x}(\tau)\big)\big]$$

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Remark: Under standard assumptions, $G = \mathcal{M}\mathbb{V}$ is upper semi-continuous if $\mathbb{V}$ is upper semi-continuous. That is, to verify optimality, continuity of $\mathbb{V}$ should suffice!

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Note: If $\mathcal{V}$ is known to be continuous, then $\mathcal{V} \in \mathbb{H}$ and things become easy.

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(4) Then $\mathfrak{V} \leq \mathcal{V} \leq \mathbb{V}$. Now apply viscosity comparison so that $\mathfrak{V} \geq \mathbb{V}$ and hence

$$\mathcal{V} = \mathbb{V} = \mathfrak{V} \qquad \text{is } \textbf{continuous}.$$

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But still: The method adapts very well. We even can show admissibility of the candidate optimal controls.

Thanks for your attention!

Belak, Christensen, Seifried (2017):
A General Verification Result for Stochastic Impulse Control Problems
To appear in *SIAM Journal on Control and Optimization*

Belak, Christensen (2017):
Utility Maximization in a Factor Model with Constant and Proportional Costs

Available at: **www.belak.ch/publications/**