Universität Trier

Optimal Stochastic Impulse Control

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Joint work with **Sören Christensen** (University of Hamburg) and **Frank Seifried** (University of Trier).

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Outline

(1) Impulse Control Problems and Quasi-Variational Inequalities

(2) Superharmonic Functions and the Stochastic Perron Method

(3) Applications and Work in Progress

The General Impulse Control Problem

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Consider an $\mathbb{R}^n$-valued **system** $X = X^{\Lambda}_{t,x}$

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\begin{aligned}
X(t) &= x, \\
\mathrm{d}X(u) &= \mu(X(u))\mathrm{d}u + \sigma(X(u))\,\mathrm{d}W(u),
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The **objective** is to maximize

$$
\mathcal{V}(t,x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[\sum_{k\in\mathbb{N}} K\big(X^{\Lambda}_{t,x}(\tau_k-), \Delta_k\big)\mathbb{1}_{\{\tau_k \le T\}} + g\big(X^{\Lambda}_{t,x}(T)\big)\Big].
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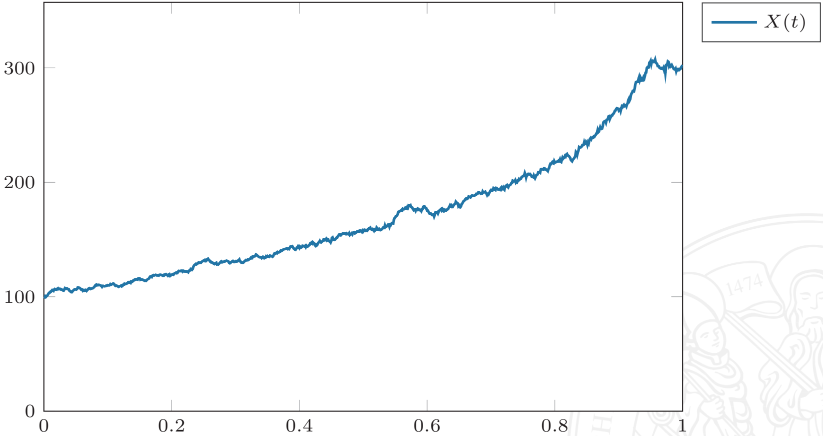
Stochastic Impulse Control Problems



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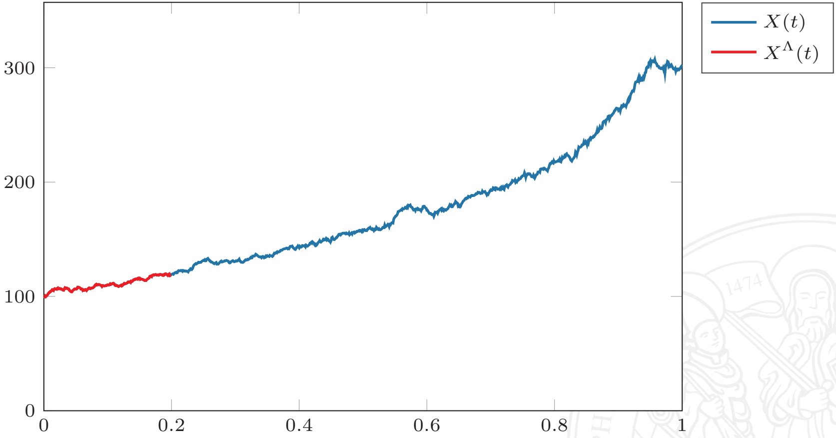

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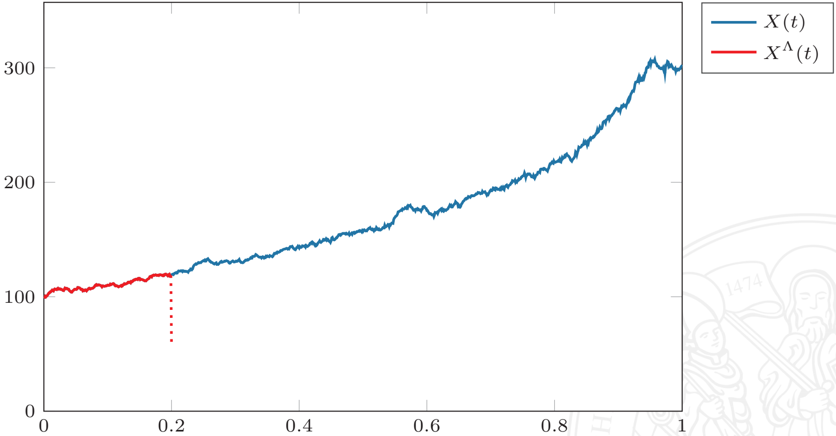

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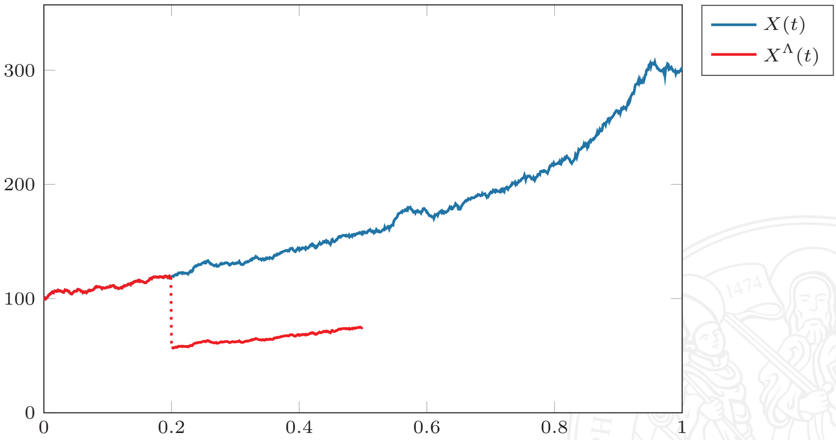

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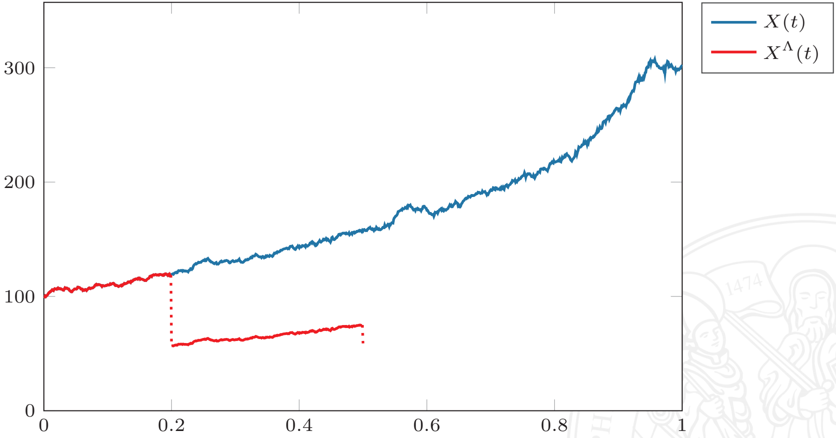

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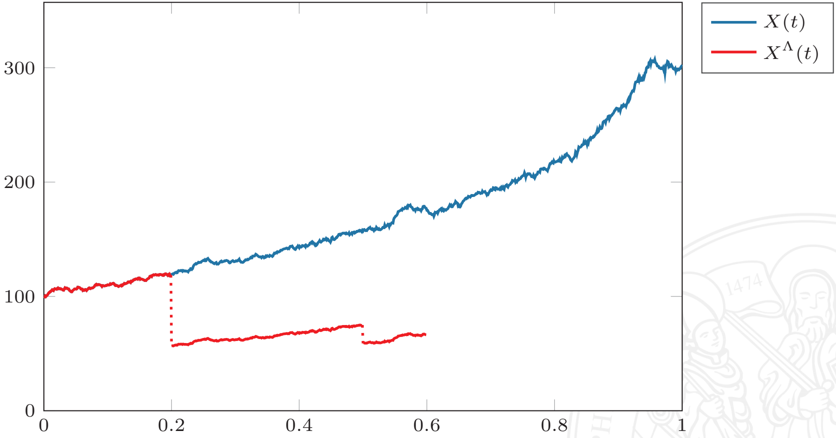

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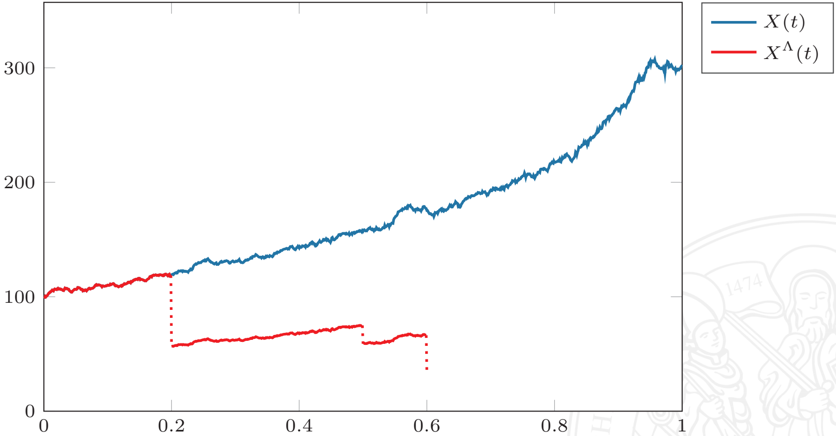

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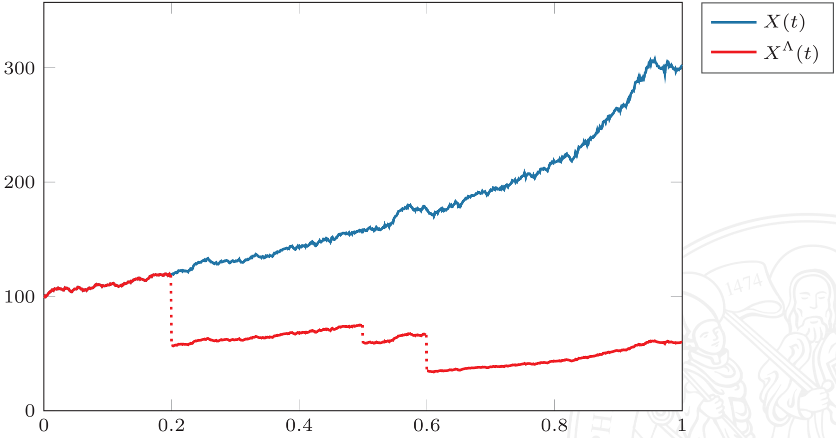

The Quasi-Variational Inequalities

The Maximum Operator

Suppose that the system is in state x at time t

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Suppose that the system is in state x at time t and the controller is **forced to make an impulse** and behaves optimally afterwards. The best immediate impulse is

$$\sup_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x, \Delta)\big) + K(x, \Delta),$$

where $Z(x)$ denotes the set of admissible impulses Δ in state x.

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$$\Delta^* = \arg\max_{\Delta \in Z(x)} \mathcal{V}\big(t, \Gamma(x,\Delta)\big) + K(x,\Delta).$$

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and equality holds if it is **not optimal** to make an impulse in the interval $[t, t+h]$.

$$\mathcal{L}\mathcal{V}(t,x) \triangleq -\partial_t \mathcal{V}(t,x) - \mu(x)\mathrm{D}_x\mathcal{V}(t,x) - \tfrac{1}{2}\,\mathrm{tr}\big[\sigma(x)\sigma(x)^\top \mathrm{D}_x^2\mathcal{V}(t,x)\big]$$

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Dividing by h and sending $h \downarrow 0$ we find that

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and equality holds **if and only if** it is not optimal to make a transaction.

The Quasi-Variational Inequalities

Thus, we have argued that $\mathcal{V}$ should solve the **quasi-variational inequalities** (QVIs)

$$\min\left\{\mathcal{L}\mathcal{V}(t,x), \mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)\right\} = 0 \qquad (t,x) \in [0,T) \times \mathbb{R}^n.$$

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Moreover, a **candidate optimal control** Λ^* is determined by the sets

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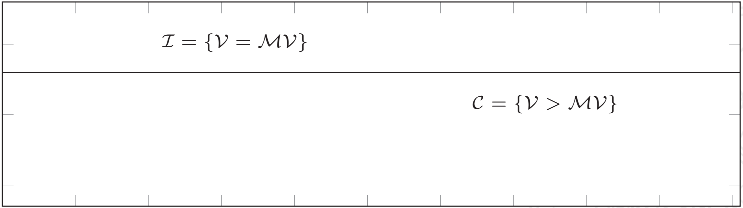

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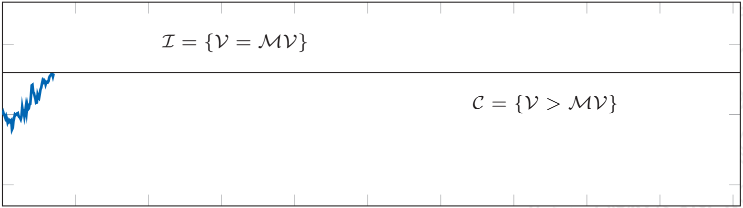

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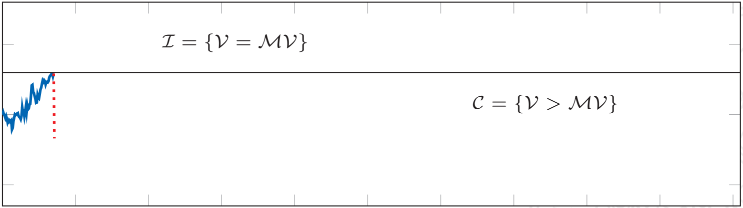

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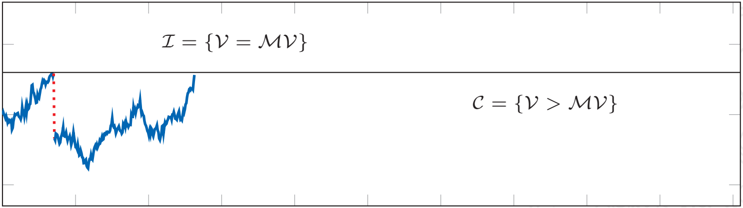

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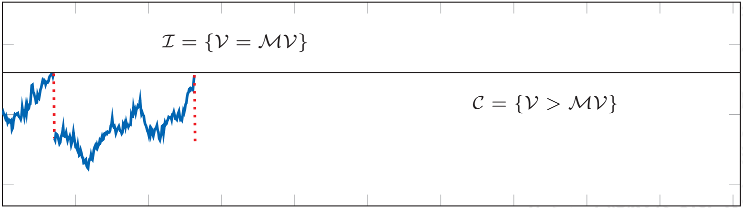

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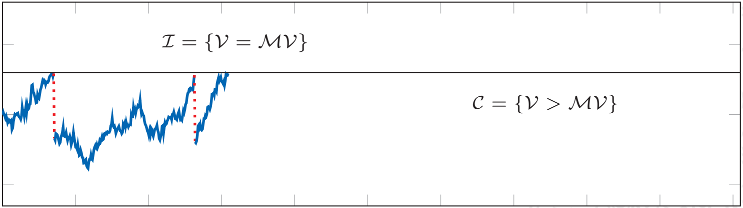

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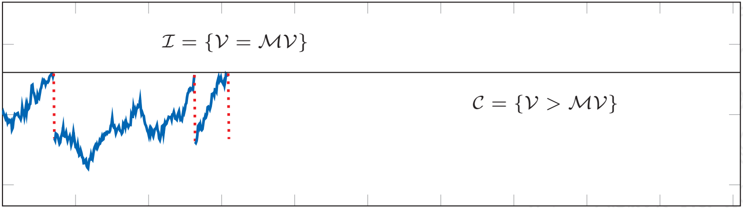

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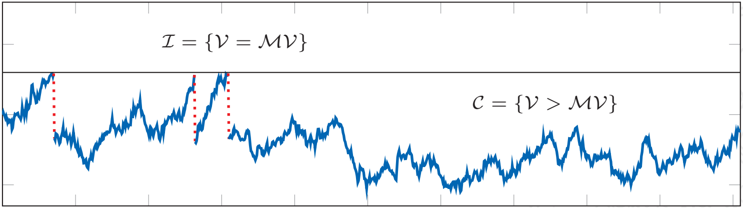

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(iii) Extend the solution to all of $[0,T] \times \mathbb{R}^n$ be setting $\mathcal{V} = \mathcal{M}\mathcal{V}$ outside of $\mathcal{C}$.

The Main Issue

So, to solve the impulse control problem, we need to solve the QVIs

$$\min\{\mathcal{L}\mathcal{V}(t,x), \mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)\} = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}^n,$$
$$\mathcal{V}(T,x) = g(x), \qquad x \in \mathbb{R}^n.$$

Once the equation is solved, determine the sets

$$\mathcal{C} \triangleq \{\mathcal{V} > \mathcal{M}\mathcal{V}\} \quad \text{and} \quad \mathcal{I} \triangleq \{\mathcal{V} = \mathcal{M}\mathcal{V}\}.$$

In practice, one proceeds as follows:

(i) Guess the set $\mathcal{C} \triangleq \{\mathcal{V} > \mathcal{M}\mathcal{V}\}$.

(ii) Solve $\mathcal{L}\mathcal{V}(t,x) = 0$ on $\mathcal{C}$.

(iii) Extend the solution to all of $[0,T] \times \mathbb{R}^n$ be setting $\mathcal{V} = \mathcal{M}\mathcal{V}$ outside of $\mathcal{C}$.

Problem: This rarely works. In higher dimensions, this is next to impossible. To verify that the solution of the QVIs coincides with $\mathcal{V}$, one needs $\mathcal{V} \in C^{1,2}$.

Superharmonic Functions and Stochastic Perron

Superharmonic Functions

We say that an integrable function $h : [0, T] \times \mathbb{R}^n$ is **superharmonic** if

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If $h \in C^{1,2}$, satisfies $\mathcal{L}h \geq 0$, and is sufficiently integrable:

$$h(t, x) = \mathbb{E}\Big[h\big(\tau, X_{t,x}(\tau)\big) + \int_t^\tau \mathcal{L}h\big(s, X_{t,x}(s)\big)\mathrm{d}s\Big]$$

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That is h is **superharmonic**.

Optimality Theorem

Let $\mathbb{H}$ be the set of all $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that

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Theorem (B., Christensen, Seifried (2016))

Suppose that $\mathcal{V}$ is continuous and $\mathcal{V} \in \mathbb{H}$. Under some technical assumptions, the control Λ^* defined in terms of

$$\{\mathcal{V} > \mathcal{M}\mathcal{V}\} \qquad \text{and} \qquad \{\mathcal{V} = \mathcal{M}\mathcal{V}\}$$

is **optimal** if it is admissible.

The stochastic Perron method:

(1) Show that $\mathbb{V} \triangleq \inf_{h\in\mathbb{H}} h \geq \mathcal{V}$ is an upper semi-continuous (viscosity) subsolution of the QVIs:

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(2) Approximate $\mathcal{V}$ from below by considering the impulse control problem $\mathcal{V}_k$ restricted to at most k impulses, and show that $\mathfrak{V} \triangleq \lim_{k\to\infty} \mathcal{V}_k$ is a lower semi-continuous (viscosity) supersolution of the QVIs:

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(4) By construction, we have $\mathfrak{V} \leq \mathcal{V} \leq \mathbb{V}$. By comparison, we have the reversed inequalities. Thus

$$\mathfrak{V} = \mathcal{V} = \mathbb{V} \qquad \text{is continuous.}$$

Sufficient Conditions for this to work

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- There exists a function $\Psi \in \mathbb{H}$ with

$$\mathbb{E}\Big[\sup_{s \in [t,T]} \big[|\Psi(s, X_{t,x}^{\Lambda})| + |\mathcal{M}\Psi(s, X_{t,x}^{\Lambda})|\big]\Big] < \infty$$

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Remark: The last two assumptions are e.g. satisfied if there exists a strict supersolution of the QVIs which grows faster at infinity than $\mathcal{V}$.

Applications and Work in Progress

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(3) Investment problem with **non-Markovian** asset prices. Joint work with Christoph Czichowsky (London).

Thanks for your attention!

Belak, Christensen, Seifried (2016):
A General Verification Result for Stochastic Impulse Control Problems
To appear in *SIAM Journal on Control and Optimization*

Belak, Christensen (2016):
Utility Maximization in a Factor Model with Constant and Proportional Costs

Available at: **www.belak.ch/publications/**