Optimal Stochastic Impulse Control

Christoph Belak
Department IV – Mathematics
University of Trier
Germany

Joint work with Sören Christensen (University of Hamburg) and Frank Seifried (University of Trier).

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(1) Impulse Control Problems and Quasi-Variational Inequalities
(2) Superharmonic Functions and the Stochastic Perron Method
(3) Applications and Work in Progress
The General Impulse Control Problem
Consider an $\mathbb{R}^n$-valued system $X = X^\Lambda_{t,x}$

$$X(t) = x,$$
$$dX(u) = \mu(X(u)) du + \sigma(X(u)) \, dW(u),$$
Consider an $\mathbb{R}^n$-valued system $X = X_{t,x}^\Lambda$ controlled by an impulse control $\Lambda = \{(\tau_k, \Delta_k)\}_{k \in \mathbb{N}}$ as follows:

\[
X(t) = x, \\
\text{d}X(u) = \mu(X(u))\text{d}u + \sigma(X(u)) \text{d}W(u), \quad u \in [\tau_k, \tau_{k+1}), \\
X(\tau_k) = \Gamma(X(\tau_k^-), \Delta_k),
\]
Consider an $\mathbb{R}^n$-valued system $X = X^\Lambda_{t,x}$ controlled by an impulse control $\Lambda = \{(\tau_k, \Delta_k)\}_{k \in \mathbb{N}}$ as follows:

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where

- the impulses $\Delta_k$ are chosen from a set $Z(X(\tau_k-)) \subset \mathbb{R}^m$, and
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where

- the impulses $\Delta_k$ are chosen from a set $Z(X(\tau_{k-})) \subset \mathbb{R}^m$, and
- the impulses do not accumulate, i.e. $\mathbb{P}[\lim_{k \to \infty} \tau_k > T] = 1$. 

The objective is to maximize $V(t,x) = \sup_{\Lambda \in \mathcal{A}} \mathbb{E} \left[ \sum_{k \in \mathbb{N}} K(X_{t,x}^\Lambda(\tau_k-), \Delta_k) 1\{\tau_k \leq T\} + g(X_{t,x}^\Lambda(T)) \right]$. 


The General Impulse Control Problem

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- the impulses $\Delta_k$ are chosen from a set $Z(X(\tau_k^-)) \subset \mathbb{R}^m$, and
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The objective is to maximize

\[
\mathcal{V}(t, x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\left[ \sum_{k \in \mathbb{N}} K\left(X_{t,x}^\Lambda(\tau_k^-), \Delta_k\right) \mathbb{1}_{\{\tau_k \leq T\}} + g\left(X_{t,x}^\Lambda(T)\right) \right].
\]
Stochastic Impulse Control Problems
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Stochastic Impulse Control Problems
The Quasi-Variational Inequalities
Suppose that the system is in state $x$ at time $t$.
The Maximum Operator

Suppose that the system is in state $x$ at time $t$ and the controller is forced to make an impulse

$$(t, \Gamma(x, \Delta)) + K(x, \Delta)$$
The Maximum Operator

Suppose that the system is in state $x$ at time $t$ and the controller is forced to make an impulse and behaves optimally afterwards.

$$V(t, \Gamma(x, \Delta)) + K(x, \Delta)$$
The Maximum Operator

Suppose that the system is in state $x$ at time $t$ and the controller is forced to make an impulse and behaves optimally afterwards. The best immediate impulse is

$$\sup_{\Delta \in Z(x)} \mathcal{V}(t, \Gamma(x, \Delta)) + K(x, \Delta),$$

where $Z(x)$ denotes the set of admissible impulses $\Delta$ in state $x$. 

The Maximum Operator

Suppose that the system is in state $x$ at time $t$ and the controller is forced to make an impulse and behaves optimally afterwards. The best immediate impulse is

$$\mathcal{M} \mathcal{V}(t, x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}(t, \Gamma(x, \Delta)) + K(x, \Delta),$$

where $Z(x)$ denotes the set of admissible impulses $\Delta$ in state $x$. 

In general, it may not be optimal to make an immediate impulse. Thus $\mathcal{V}(t, x) \geq \mathcal{M} \mathcal{V}(t, x)$.

If $\mathcal{V}(t, x) = \mathcal{M} \mathcal{V}(t, x)$, an impulse is expected to be optimal. A candidate is $\Delta^* = \operatorname{arg\ max}_{\Delta \in Z(x)} \mathcal{V}(t, \Gamma(x, \Delta)) + K(x, \Delta)$. 

Suppose that the system is in state $x$ at time $t$ and the controller is forced to make an impulse and behaves optimally afterwards. The best immediate impulse is

$$\mathcal{MV}(t, x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}(t, \Gamma(x, \Delta)) + K(x, \Delta),$$

where $Z(x)$ denotes the set of admissible impulses $\Delta$ in state $x$.

In general, it may not be optimal to make an immediate impulse. Thus

$$\mathcal{V}(t, x) \geq \mathcal{MV}(t, x).$$
Suppose that the system is in state $x$ at time $t$ and the controller is forced to make an impulse and behaves optimally afterwards. The best immediate impulse is

$$\mathcal{M} V(t, x) \triangleq \sup_{\Delta \in Z(x)} \mathcal{V}(t, \Gamma(x, \Delta)) + K(x, \Delta),$$

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$$\Delta^* = \arg \max_{\Delta \in Z(x)} \mathcal{V}(t, \Gamma(x, \Delta)) + K(x, \Delta).$$
The Infinitesimal Generator

Let the state process $X$ run uncontrolled for a positive amount of time, say on the interval $[t, t + h]$. 
Let the state process $X$ run \textbf{uncontrolled} for a positive amount of time, say on the interval $[t, t + h]$. Time consistency lets us expect that

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\mathcal{V}(t, x) \geq \mathbb{E}[\mathcal{V}(t + h, X_{t,x}(t + h))]
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and equality holds if it is **not optimal** to make an impulse in the interval \([t, t + h]\).
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= \mathbb{E}[\mathcal{V}(t, x) - \int_t^{t+h} \mathcal{L}\mathcal{V}(u, X_{t,x}(u))du]
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and equality holds if it is **not optimal** to make an impulse in the interval $[t, t + h]$. 

$$
\mathcal{L}\mathcal{V}(t, x) \triangleq -\partial_t \mathcal{V}(t, x) - \mu(x)D_x \mathcal{V}(t, x) - \frac{1}{2} \text{tr}[\sigma(x)\sigma(x)^\top D_x^2 \mathcal{V}(t, x)]
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Dividing by $h$ and sending $h \downarrow 0$ we find that

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\mathcal{L}\mathcal{V}(t, x) \triangleq -\partial_t \mathcal{V}(t, x) - \mu(x)D_x\mathcal{V}(t, x) - \frac{1}{2} \text{tr} \left[ \sigma(x)\sigma(x)^\top D_x^2\mathcal{V}(t, x) \right] \geq 0
$$
Let the state process $X$ run **uncontrolled** for a positive amount of time, say on the interval $[t, t + h]$. Time consistency lets us expect that

$$V(t, x) \geq \mathbb{E}[V(t + h, X_{t,x}(t + h))]$$

$$= \mathbb{E}[V(t, x) - \int_t^{t+h} \mathcal{L}V(u, X_{t,x}(u))du]$$

and equality holds if it is **not optimal** to make an impulse in the interval $[t, t + h]$.

Dividing by $h$ and sending $h \downarrow 0$ we find that

$$\mathcal{L}V(t, x) \triangleq -\partial_t V(t, x) - \mu(x)D_x V(t, x) - \frac{1}{2} \text{tr} \left[ \sigma(x)\sigma(x)^\top D_x^2 V(t, x) \right] \geq 0$$

and equality holds **if and only if** it is not optimal to make a transaction.
Thus, we have argued that $\mathcal{V}$ should solve the quasi-variational inequalities (QVIs)

$$\min \{ \mathcal{L}\mathcal{V}(t, x), \mathcal{V}(t, x) - \mathcal{M}\mathcal{V}(t, x) \} = 0 \quad (t, x) \in [0, T) \times \mathbb{R}^n.$$
The Quasi-Variational Inequalities

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Moreover, a candidate optimal control $\Lambda^*$ is determined by the sets

$$\mathcal{C} \triangleq \{ \mathcal{V} > \mathcal{M}\mathcal{V} \} \quad \text{and} \quad \mathcal{I} \triangleq \{ \mathcal{V} = \mathcal{M}\mathcal{V} \}.$$
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Thus, we have argued that $V$ should solve the \textit{quasi-variational inequalities} (QVIs)

$$\min \{ L V(t, x), V(t, x) - M V(t, x) \} = 0 \quad (t, x) \in [0, T) \times \mathbb{R}^n. $$

Moreover, a \textit{candidate optimal control} $\Lambda^*$ is determined by the sets

$$C \triangleq \{ V > M V \} \quad \text{and} \quad I \triangleq \{ V = M V \}. $$
Thus, we have argued that \( V \) should solve the **quasi-variational inequalities** (QVIs)

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\min \{ \mathcal{L}V(t, x), V(t, x) - M V(t, x) \} = 0 \quad (t, x) \in [0, T) \times \mathbb{R}^n.
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Moreover, a **candidate optimal control** \( \Lambda^* \) is determined by the sets

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Thus, we have argued that $\mathcal{V}$ should solve the **quasi-variational inequalities** (QVIs)

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Moreover, a **candidate optimal control** $\Lambda^*$ is determined by the sets

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The Quasi-Variational Inequalities

Thus, we have argued that \( V \) should solve the **quasi-variational inequalities** (QVIs)

\[
\min \{ \mathcal{L}V(t, x), V(t, x) - \mathcal{M}V(t, x) \} = 0 \quad (t, x) \in [0, T) \times \mathbb{R}^n.
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Moreover, a **candidate optimal control** \( \Lambda^* \) is determined by the sets

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\mathcal{C} \triangleq \{ V > \mathcal{M}V \} \quad \text{and} \quad \mathcal{I} \triangleq \{ V = \mathcal{M}V \}.
\]
So, to solve the impulse control problem, we need to solve the QVIs

\[
\min \{ \mathcal{L}V(t, x), V(t, x) - \mathcal{M}V(t, x) \} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n,
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V(T, x) = g(x), \quad x \in \mathbb{R}^n.
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\mathcal{V}(T, x) = g(x), \quad x \in \mathbb{R}^n.
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Once the equation is solved, determine the sets
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So, to solve the impulse control problem, we need to solve the QVIs

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\min \{ \mathcal{L}V(t, x), V(t, x) - M V(t, x) \} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n,
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V(T, x) = g(x), \quad x \in \mathbb{R}^n.
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**In practice**, one proceeds as follows:
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**In practice**, one proceeds as follows:

(i) Guess the set $$\mathcal{C} \triangleq \{V > \mathcal{M}V\}.$$
So, to solve the impulse control problem, we need to solve the QVIs

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**In practice**, one proceeds as follows:

(i) Guess the set \(\mathcal{C} \triangleq \{V > \mathcal{M}V\}\).

(ii) Solve \(\mathcal{L}V(t, x) = 0\) on \(\mathcal{C}\).
So, to solve the impulse control problem, we need to solve the QVIs

$$\min\{\mathcal{L}V(t, x), V(t, x) - \mathcal{M}V(t, x)\} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n,$$

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**In practice**, one proceeds as follows:

(i) Guess the set \( \mathcal{C} \triangleq \{V > \mathcal{M}V\} \).

(ii) Solve \( \mathcal{L}V(t, x) = 0 \) on \( \mathcal{C} \).

(iii) Extend the solution to all of \([0, T] \times \mathbb{R}^n\) by setting \( V = \mathcal{M}V \) outside of \( \mathcal{C} \).
So, to solve the impulse control problem, we need to solve the QVIs

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\min \{ \mathcal{L} \mathcal{V}(t, x), \mathcal{V}(t, x) - \mathcal{M} \mathcal{V}(t, x) \} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \\
\mathcal{V}(T, x) = g(x), \quad x \in \mathbb{R}^n.
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Once the equation is solved, determine the sets

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\mathcal{C} \triangleq \{ \mathcal{V} > \mathcal{M} \mathcal{V} \} \quad \text{and} \quad \mathcal{I} \triangleq \{ \mathcal{V} = \mathcal{M} \mathcal{V} \}.
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**In practice**, one proceeds as follows:

(i) Guess the set \( \mathcal{C} \triangleq \{ \mathcal{V} > \mathcal{M} \mathcal{V} \} \).

(ii) Solve \( \mathcal{L} \mathcal{V}(t, x) = 0 \) on \( \mathcal{C} \).

(iii) Extend the solution to all of \([0, T] \times \mathbb{R}^n\) by setting \( \mathcal{V} = \mathcal{M} \mathcal{V} \) outside of \( \mathcal{C} \).

**Problem**: This rarely works. In higher dimensions, this is next to impossible. To verify that the solution of the QVIs coincides with \( \mathcal{V} \), one needs \( \mathcal{V} \in C^{1,2} \).
Superharmonic Functions and Stochastic Perron
We say that an integrable function $h : [0, T] \times \mathbb{R}^n$ is superharmonic if

$$h(t, x) \geq \mathbb{E}\left[h(\tau, X_{t,x}(\tau))\right]$$

for any choice of $(t, x)$ and $\tau$. 
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for any choice of $(t, x)$ and $\tau$.

If $h \in C^{1,2}$, satisfies $\mathcal{L}h \geq 0$, and is sufficiently integrable:

$$h(t, x) = \mathbb{E}\left[ h(\tau, X_{t,x}(\tau)) + \int_t^\tau \mathcal{L}h(s, X_{t,x}(s)) \, ds \right]$$
Superharmonic Functions

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If $h \in C^{1,2}$, satisfies $\mathcal{L}h \geq 0$, and is sufficiently integrable:

$$h(t, x) = \mathbb{E} \left[ h(\tau, X_{t,x}(\tau)) + \int_t^\tau \mathcal{L}h(s, X_{t,x}(s)) ds \right] \geq \mathbb{E} \left[ h(\tau, X_{t,x}(\tau)) \right].$$

That is $h$ is superharmonic.
Optimality Theorem

Let $\mathcal{H}$ be the set of all $h : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $h$ is superharmonic,
Optimality Theorem

Let $\mathcal{H}$ be the set of all $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that

- $h$ is superharmonic,
- $h$ is non-increasing in the direction of impulses, i.e. $h \geq \mathcal{M}h$, 

It is straightforward to see that $h \geq V$ for all $h \in \mathcal{H}$.

Theorem (B., Christensen, Seifried (2016))

Suppose that $V$ is continuous and $V \in \mathcal{H}$. Under some technical assumptions, the control $\Lambda^*$ defined in terms of $\{V > MV\}$ and $\{V = MV\}$ is optimal if it is admissible.
Let $\mathcal{H}$ be the set of all $h : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $h$ is superharmonic,
- $h$ is non-increasing in the direction of impulses, i.e. $h \geq \mathcal{M}h$,
- $h$ satisfies the terminal condition $h(T, \cdot) \geq g$,
Optimality Theorem

Let $\mathcal{H}$ be the set of all $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that

- $h$ is superharmonic,
- $h$ is non-increasing in the direction of impulses, i.e. $h \geq \mathcal{M}h$,
- $h$ satisfies the terminal condition $h(T, \cdot) \geq g$,
- $h$ is upper semi-continuous.
Optimality Theorem

Let $\mathcal{H}$ be the set of all $h : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $h$ is superharmonic,
- $h$ is non-increasing in the direction of impulses, i.e. $h \geq Mh$,
- $h$ satisfies the terminal condition $h(T, \cdot) \geq g$,
- $h$ is upper semi-continuous.

It is straightforward to see that $h \geq \mathcal{V}$ for all $h \in \mathcal{H}$. 
Optimality Theorem

Let $\mathbb{H}$ be the set of all $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that

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Theorem (B., Christensen, Seifried (2016))

Suppose that $\mathcal{V}$ is continuous and $\mathcal{V} \in \mathbb{H}$. Under some technical assumptions, the control $\Lambda^*$ defined in terms of

$$\{\mathcal{V} > \mathcal{M}\mathcal{V}\} \quad \text{and} \quad \{\mathcal{V} = \mathcal{M}\mathcal{V}\}$$

is optimal if it is admissible.
The stochastic Perron method:

(1) Show that $V \triangleq \inf_{h \in \mathcal{H}} h \geq \mathcal{V}$ is an upper semi-continuous (viscosity) subsolution of the QVIs:

$$\min\{\mathcal{L}V(t, x), V(t, x) - \mathcal{M}V(t, x)\} \leq 0.$$
The Stochastic Perron Method

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(1) Show that \( \mathcal{V} \triangleq \inf_{h \in \mathcal{H}} h \geq \mathcal{V} \) is an upper semi-continuous (viscosity) subsolution of the QVIs:

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(2) Approximate \( \mathcal{V} \) from below by considering the impulse control problem \( \mathcal{V}_k \) restricted to at most \( k \) impulses, and show that \( \mathcal{V} \triangleq \lim_{k \to \infty} \mathcal{V}_k \) is a lower semi-continuous (viscosity) supersolution of the QVIs:

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The Stochastic Perron Method

**The stochastic Perron method:**

(1) Show that $V \triangleq \inf_{h \in H} h \geq \mathcal{V}$ is an upper semi-continuous (viscosity) subsolution of the QVIs:

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(2) Approximate $V$ from below by considering the impulse control problem $\mathcal{V}_k$ restricted to at most $k$ impulses, and show that $\mathcal{V} \triangleq \lim_{k \to \infty} \mathcal{V}_k$ is a lower semi-continuous (viscosity) supersolution of the QVIs:

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(3) Prove a comparison principle for the QVIs: If $u$ is a subsolution and $v$ is a supersolution of the QVIs with $u(T, \cdot) \leq v(T, \cdot)$, then $u \leq v$. 

By construction, we have $V \leq \mathcal{V} \leq \mathcal{V}$. By comparison, we have the reversed inequalities. Thus $V = \mathcal{V} = \mathcal{V}$ is continuous.
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Sufficient Conditions for this to work

The stochastic Perron method works, e.g., under the following assumptions:

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Applications and Work in Progress
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(3) Investment problem with non-Markovian asset prices. Joint work with Christoph Czichowsky (London).
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