Universität Trier

Utility Maximization with Constant and Proportional Costs

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Joint work with **Sören Christensen** (Göteborg).

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Market Model and Problem Formulation

The Market Model and Transaction Costs

The **financial market** consists of n assets $P = (P^1, P^2, \ldots, P^n)$ with prices

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where $\gamma_i \in (0,1)$ and $K_i > 0$ for $i = 2, \dots, n$.

Solvency and Trading Strategies

A portfolio $x \in \mathbb{R}^n$ is **solvent** if it has a positive **liquidation value** $L(x)$, i.e.

$$L(x) \triangleq x_1 + \sum_{i=2}^{n} [x_i - \gamma_i |x_i| - K_i] > 0.$$

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In the above,

- τ_Λ denotes the **bankruptcy time** of the strategy Λ, i.e. the first time X^Λ leaves the solvency region $\mathcal{S}$, and
- $\mathcal{A}(t, x, y)$ denotes the set of admissible strategies corresponding to an initial state of (t, x, y).

Related Literature

In the existing literature, there are three different approaches:

- **Classical Verification**: Eastham and Hastings (1988), Korn (1998), Bielecki and Pliska (2000), Liu (2004), ...
- **Viscosity Characterization**: Oksendal and Sulem (2002), ...
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Note: To the best of our knowledge, not even in the Black-Scholes case with log or power utility, a rigorous existence result for optimal strategies is known.

The Candidate Optimal Strategy

The Quasi-Variational Inequalities

The Bellman principle suggests that $\mathcal{V}$ can be linked to the **quasi-variational inequalities** (QVIs)

$$\min\Big\{-\partial_t \mathcal{V}(t,x,y) - \mathcal{L}\mathcal{V}(t,x,y), \mathcal{V}(t,x,y) - \mathcal{M}\mathcal{V}(t,x,y)\Big\} = 0$$

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Note: $\mathcal{M}\mathcal{V}(t,x,y)$ can be thought of as the highest expected utility that can be achieved if the investor is **forced to make a transaction** at time t.

A Candidate Optimal Control

A candidate optimal control can be defined in terms of the **continuation region** $\mathcal{C}$ and the **intervention region** $\mathcal{I}$ given by

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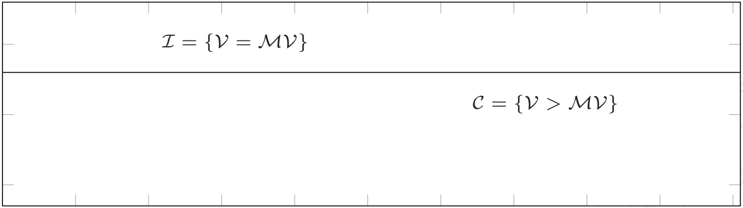

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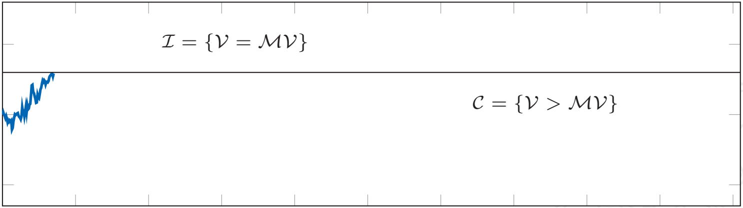

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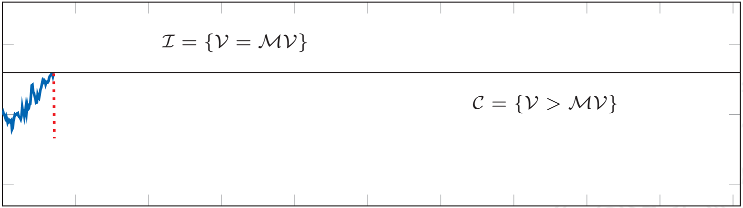

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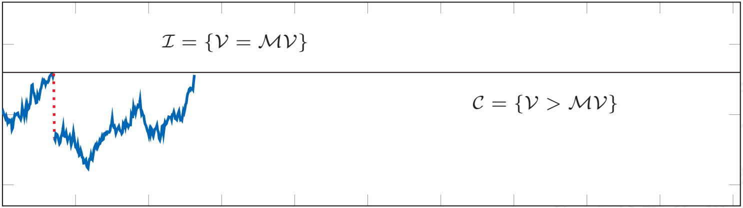

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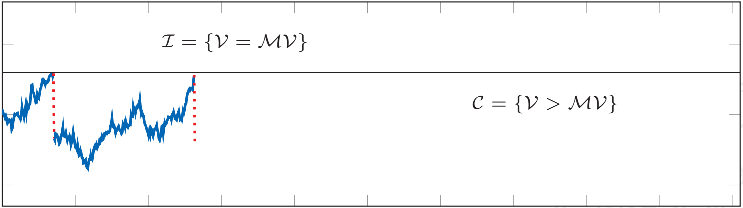

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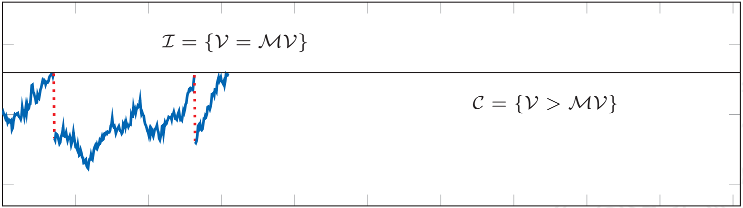

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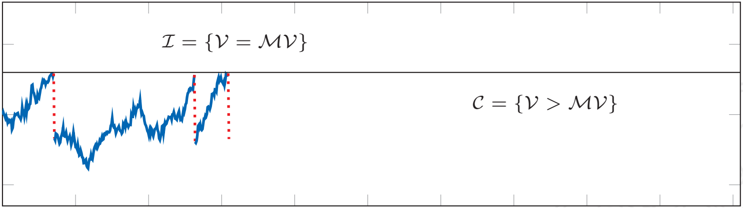

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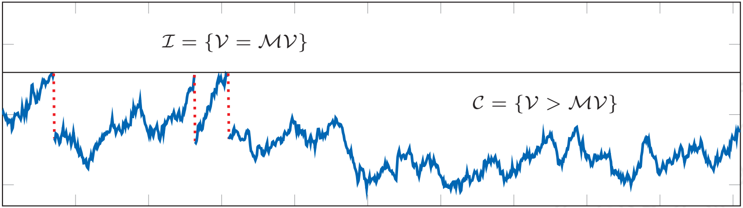

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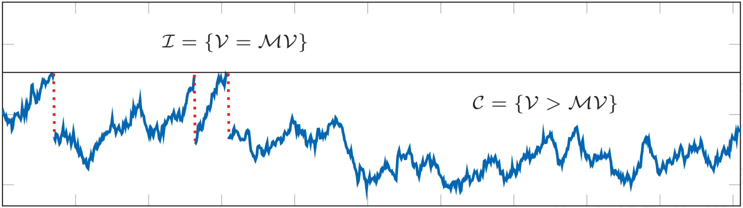

Problem: To verify optimality we need sufficient regularity to apply Itô's formula, but it seems unlikely that the QVIs admit a $C^{1,2}$-solution.

Verification via Superharmonic Functions

A Formal Optimal Stopping Problem

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Remark: $G = \mathcal{M}\mathcal{V}$ is upper semi-continuous if $\mathcal{V} = \mathbb{V}$ is upper semi-continuous. That is, to verify optimality, continuity of $\mathbb{V}$ should suffice!

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Verification Theorem (Belak and Christensen (2016))

Suppose that $\mathbb{V}$ is continuous. Then $\mathbb{V} = \mathcal{V}$ and the candidate optimal control Λ^* defined in terms of the sets

$$\mathcal{C} = \{\mathbb{V} > \mathcal{M}\mathbb{V}\} \quad \text{and} \quad \mathcal{I} = \{\mathbb{V} = \mathcal{M}\mathbb{V}\}.$$

is admissible and optimal.

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(4) By **viscosity comparison**: $\mathbb{V} \leq \mathbb{V}_*$, i.e. $\mathbb{V}$ is continuous.

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- The approach **generalizes** to a big class of general impulse control problems (see Belak, Christensen, and Seifried (2016)).

Thanks for your attention!

Belak and Christensen (2016):
Utility Maximization in a Factor Model with Constant and Proportional Costs

Belak, Christensen, and Seifried (2016):
A General Verification Result for Stochastic Impulse Control Problems

Available at: **www.belak.ch/publications/**