Universität Trier

Portfolio Optimization with Transaction Costs

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Joint work with **Sören Christensen** (Göteborg) and **Frank T. Seifried** (Trier).

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(6) Work in progress: We **generalize** our method to more general markets and more general optimization problems.

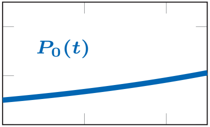
The Market Model and the Optimization Problem

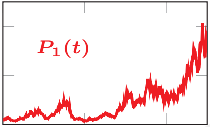
The Financial Market

The financial market consists of a money market account P_0 and a stock P_1 with

$$P_0(t) = p_0 e^{rt} \quad \text{and} \quad P_1(t) = p_1 e^{(\mu - \sigma^2/2)t + \sigma W_t},$$

where p_0, p_1, r, μ, σ are constants and W is a Brownian motion.





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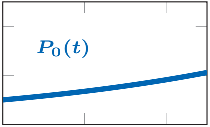

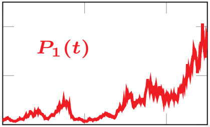

In differential form, the price dynamics can be written as

$$\begin{aligned} dP_0(t) &= rP_0(t)dt, & P_0(0) &= p_0, \\ dP_1(t) &= \mu P_1(t)dt + \sigma P_1(t)dW(t), & P_1(0) &= p_1. \end{aligned}$$

Constant and Proportional Transaction Costs

Upon buying or selling shares of the stock worth Δ units of money, the investor has to pay **transaction costs** of size

$$\gamma|\Delta| + K.$$

We refer to $\gamma \in (0,1)$ as the **proportional cost** component and to $K > 0$ as the **constant cost** component.

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Trading Strategies and Wealth Dynamics

Trading strategies are modeled as sequences $(\tau_1, \Delta_1), (\tau_2, \Delta_2), \dots$ consisting of **trading dates** τ_j and **transaction volumes** Δ_j, where $j \in \mathbb{N}$.

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The **wealth** in the money market account X_0 and the stock account X_1 evolve as

$$\begin{aligned} \mathrm{d}X_0(t) &= rX_0(t)\mathrm{d}t & X_0(0) &= x_0, \\ \mathrm{d}X_1(t) &= \mu X_1(t)\mathrm{d}t + \sigma X_1(t)\mathrm{d}W(t) & X_1(0) &= x_1. \end{aligned}$$

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$$\begin{aligned} \mathrm{d}X_0(t) &= rX_0(t)\mathrm{d}t - \textstyle\sum_{j=1}^{\infty}[\Delta_j + \gamma|\Delta_j| + K]\mathbb{1}_{\{\tau_j = t\}}, \qquad & X_0(0) &= x_0, \\ \mathrm{d}X_1(t) &= \mu X_1(t)\mathrm{d}t + \sigma X_1(t)\mathrm{d}W(t) + \textstyle\sum_{j=1}^{\infty}\Delta_j \mathbb{1}_{\{\tau_j = t\}}, & X_1(0) &= x_1. \end{aligned}$$

Liquidation Value and Admissibility

The **liquidation value** of a portfolio position $x = (x_0, x_1)$ is given by

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We define the **solvency region** as

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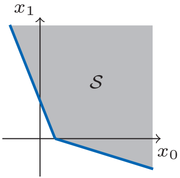

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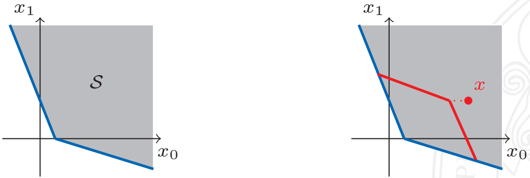
A trading strategy $\Lambda = \{(\tau_j, \Delta_j)\}_{j\in\mathbb{N}}$ is called **admissible**, if

$$\big(X_0^\Lambda(\tau_j-) - \Delta_j - K - \gamma|\Delta_j|, X_1^\Lambda(\tau_j-) + \Delta_j\big) \in \overline{\mathcal{S}} \qquad \text{for all } j \in \mathbb{N}.$$



The Optimization Problem

Our aim is to maximize the **expected utility** of the liquidation value at time T, i.e.

$$\mathcal{V}(t,x) \triangleq \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\big[U\big(L(X^{\Lambda}_{t,x}(\tau_{\mathcal{S}} \wedge T))\big)\big], \qquad (t,x) \in [0,T) \times \mathcal{S}.$$

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- $\mathcal{A}(t,x)$ denotes the set of all **admissible** trading strategies.

Derivation of the Quasi-Variational Inequalities

The Maximum Operator

Suppose that the investor has a wealth of $x = (x_0, x_1)$ at time time t and that she is **forced to make a transaction**.

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$$\sup_{\Delta \in Z(x)} \mathcal{V}\big(t, x_0 - \Delta - K - \gamma|\Delta|, x_1 + \Delta\big),$$

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$$\Delta^* = \arg\max_{\Delta \in Z(x)} \mathcal{V}\big(t, x_0 - \Delta - K - \gamma|\Delta|, x_1 + \Delta\big).$$

The Differential Operator

Time consistency lets us expect that

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Dividing by h and sending $h \downarrow 0$ we find that

$$\mathcal{L}\mathcal{V}(t,x) \geq 0$$

and equality holds **if and only if** it is not optimal to make a transaction.

The Quasi-Variational Inequalities

Putting the pieces together, we have argued that $\mathcal{V}$ should solve the **quasi-variational inequalities** (QVIs)

$$\min\{\mathcal{L}\mathcal{V}(t,x), \mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)\} = 0 \qquad (t,x) \in [0,T) \times \mathcal{S}.$$

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Moreover, a **candidate optimal trading strategy** Λ^* is determined by the sets

$$\mathcal{C} \triangleq \{(t,x) \in [0,T) \times \mathcal{S} \; : \; \mathcal{V}(t,x) > \mathcal{M}\mathcal{V}(t,x)\},$$
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Problem: It is unlikely that $\mathcal{V}$ solves the QVIs in the classical sense.

Our Approach: Uniqueness implies Existence

First Observation: Superharmonicity of $\mathcal{V}$

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Answer: In essence, just to argue that

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Idea: Replace the requirement $\mathcal{L}\mathcal{V}(t,x) \geq 0$ by the requirement

$$\mathcal{V}(t,x) \geq \mathbb{E}\big[\mathcal{V}\big(\theta, \bar{X}(\theta)\big)\big] \quad \text{for any stopping time } \theta.$$

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Superharmonic Characterization of $\mathcal{V}$

Let $\mathbb{V}$ denote the pointwise infimum of the functions in $\mathbb{H}$, i.e.

$$\mathbb{V}(t,x) = \inf\big\{h(t,x) \ : \ h \in \mathbb{H}\big\}.$$

If $\mathbb{V}$ is continuous, then $\mathbb{V} = \mathcal{V}$ and the candidate strategy Λ^* is optimal.

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Comparison Principle for the QVIs

Let u be an upper semi-continuous viscosity supersolution and v be a lower semi-continuous viscosity subsolution of the QVIs such that $u(T, \cdot) \leq v(T, \cdot)$. Then $u \leq v$ everywhere.

Viscosity Characterization

Viscosity Characterization of $\mathbb{V}$

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- No measurable selection arguments/weak formulation/path space assumption to obtain the **viscosity characterization**.

Where to go from here?

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Thank you very much for your attention!