Universität Trier

Pricing Contingent Claims under Jump Uncertainty

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Literature: Hua/Wilmott (1997), Mönnig (2012).

Wealth Dynamics

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Here,

- $\zeta_1 = (\zeta_1^1, \ldots, \zeta_1^n)$ is the **strategy** used before the jump,
- $\vartheta = (\tau, \beta)$ is the **jump** with $\tau \in [0, T] \cup \{\infty\}$ and $\beta \in [\beta_D(\tau), \beta_U(\tau)]$,

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- $\zeta_0^\vartheta = (\zeta_0^{1,\vartheta}, \dots, \zeta_0^{n,\vartheta})$ is the **strategy** used after the jump ϑ.

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The Jump Uncertainty Price

A **contingent claim** $\xi(P^0(T), P(T))$ is an $\mathcal{F}(T)$-measurable, non-negative random variable, Lipschitz continuous in the prices (P^0, P) of the underlyings.

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Definition: Jump Uncertainty Price

The **jump uncertainty price** $\mathcal{V}_1$ is defined as

$$\mathcal{V}_1 \triangleq \inf\Big\{x \geq 0 : \exists(\zeta_1, \zeta_0) \text{ s.t. } X_{t,x}^{\zeta_1,\zeta_0,\vartheta}(T) \geq \xi(P^0(T), P^\vartheta(T)) \text{ for every jump } \vartheta\Big\}.$$

In other words, $\mathcal{V}_1$ is the smallest initial wealth that is required to superhedge the claim in **every jump scenario**.

The Crash Constraint

The Main Idea

In order to superhedge the claim, the trader has to ensure that the wealth after a jump dominates the price in the jump-free market.

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Mathematically, this means that

$$H\big(\tau, X_x^{\zeta_1}(\tau-), \zeta_1(\tau)\big) \geq 0 \qquad \text{for all stopping times } \tau,$$

where the **jump constraint** H is defined as

$$H(t,x,\zeta) \triangleq x - \sup_{\beta \in [\beta_D(t), \beta_U(t)]} \Big[\mathcal{V}_0(t,\beta) - \zeta^\top \beta\Big].$$

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$\mathcal{V}_1$ is **well-defined** if and only if there are x, ζ_1 satisfying the constraint.

BSDE Characterization

Theorem (BSDE Characterization of $\mathcal{V}_1$)

The jump uncertainty price is given by $\mathcal{V}_1 = X(0)$, where (X, ζ_1) denotes the **minimal supersolution** of the BSDE

$$\mathrm{d}X(t) = \big[r(t)X(t) + \zeta_1(t)^\top \sigma(t)\theta(t)\big]\,\mathrm{d}t + \zeta_1(t)^\top \sigma(t)\,\mathrm{d}W(t),$$
$$X(T) \geq \xi\big(P^0(T), P(T)\big).$$

under the **constraint**

$$H\big(t, X(t), \zeta_1(t)\big) \geq 0 \qquad \text{for all } t \in [0,T], \mathbb{P}\text{-a.s.}$$

Literature: Peng (1999), Kharroubi/Ma/Pham/Zhang (2010).

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In the case of **Markovian price dynamics**, $\mathcal{V}_1 = \mathcal{V}_1(t, p_0, p)$

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In the case of **Markovian price dynamics**, $\mathcal{V}_1 = \mathcal{V}_1(t, p_0, p)$ is the unique continuous viscosity solution of

$$\min\Big\{-\partial_t \mathcal{V}_1 - \mathcal{L}\mathcal{V}_1 + r(t, p_0, p)\mathcal{V}_1, H(t, p_0, p, \mathcal{V}_1, \operatorname{diag}(p) D_p \mathcal{V}_1)\Big\} = 0,$$

Remark: $\mathcal{L}$ is the generator of (P^0, P) under the risk neutral measure.

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satisfying the **terminal condition**

$$\min\Big\{\mathcal{V}_1 - \xi(p_0, p), H(T, p_0, p, \mathcal{V}_1, \mathrm{diag}(p) D_p \mathcal{V}_1)\Big\} = 0.$$

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Black-Scholes with Jumps

Consider a **Black-Scholes market** with constant minimum and maximum jump sizes β_D and β_U, and let ξ be a **European call** with strike price K.

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The terminal condition is given explicitly as

$$\mathcal{V}_1(T-, p) = [p - K]\mathbb{1}_{\{p \geq L\}} + \alpha_D p^{-1/\beta_D}\mathbb{1}_{\{p<L\}} + \alpha_U p^{-1/\beta_U}\mathbb{1}_{\{p \geq L\}}.$$

Remark: The adjusted strike is given by $L \triangleq K/(1 + \beta_U)(1 + \beta_D)$.

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The jump uncertainty price $\mathcal{V}_1$ is given **explicitly** as

$$\begin{aligned}\mathcal{V}_1(t,p) = p\Phi(d_1(L)) - Ke^{-r(T-t)}\Phi(d_2(L)) \\ + \alpha_D\eta_D(t)p^{-1/\beta_D}\Phi(-d_2(L) + \sigma\sqrt{T-t}/\beta_D) \\ + \alpha_U\eta_U(t)p^{-1/\beta_U}\Phi(d_2(L) - \sigma\sqrt{T-t}/\beta_U)\end{aligned}$$

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Numerics: The Jump Uncertainty Price

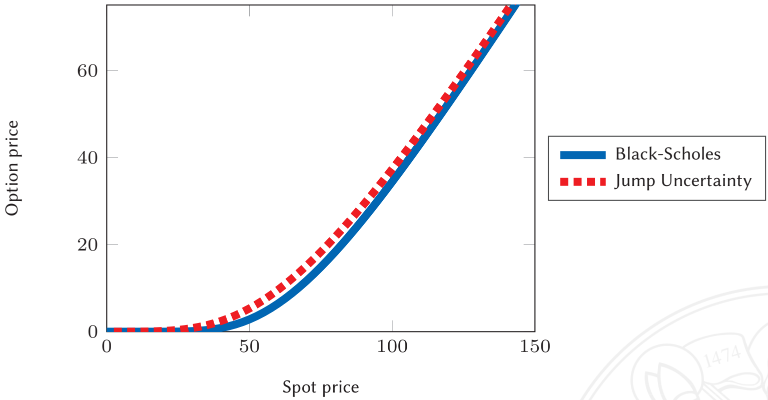

Figure: Jump uncertainty price for two-sided jumps; $\beta_D = -0.25$, $\beta_U = 0.25$.

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Thank you very much for your attention!