Worst-Case Portfolio Optimization
Transaction Costs and Bubbles

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Overview

(1) The Basics of Optimal Investment
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(2) The Worst-Case Approach to Crash Modeling
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(3) Portfolio Optimization with Transaction Costs
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(2) The Worst-Case Approach to Crash Modeling

(3) Portfolio Optimization with Transaction Costs
How do we invest our money optimally in a financial market?
How do we invest our money **optimally** in a financial market?
We need a model for the financial market:
The Building Blocks of our Models

We need a model for the financial market:
- Assets, asset prices and wealth dynamics

- Investment objective
- Risk preferences

In this talk, we are interested in an investor
- who wants to make a long-term investment,
- aims at maximizing her wealth,
- is risk-averse,
- acknowledges the possibility of severe market crashes,
- faces transaction costs.
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Example: The Merton Model (1969)

**The financial market**: One bond/bank account, one stock.
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Here: \( x \) initial wealth, \( X_t \) wealth at time \( t \)
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The investor’s preferences:

\[
\sup_{\pi} \mathbb{E} \left[ U_p (X_T^\pi) \right],
\]

\[
\begin{array}{c}
\text{Bond price} \\
\text{Stock price}
\end{array}
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The investor’s preferences:

\[ \sup \mathbb{E} \left[ U_p \left( X^\pi_T \right) \right], \]

where \( U_p(x) = x^p/p, p < 1 \), denotes the investor’s utility function.
The optimal strategy $\pi_M$ in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1 - p)\sigma^2}.$$
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Idea: Let’s add a jump component to the stock price dynamics!

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where \( N \) is a compensated Poisson process.
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**Drawbacks of this approach:**

- The optimal strategy does not depend on the investment horizon.
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Drawbacks of this approach:

- The optimal strategy does not depend on the investment horizon.
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- It is hard to estimate the crash parameters.
Instead of assigning probabilities to crashes, only assume an upper bound on the number and the size of crashes.
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We assume that the stock may crash by an unknown fraction \( \beta \in [0, \beta^*] \) at an unknown stopping time \( \tau \in [0, T] \cup \{\infty\} \).
Instead of assigning probabilities to crashes, only assume an upper bound on the number and the size of crashes.

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$$X^\pi_\tau = X^\pi_{\tau^{-}} - \beta \pi_\tau X^\pi_{\tau^{-}} \quad \text{on } \{\tau < \infty\}.$$
The Worst-Case Approach

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$$X_\tau^\pi = X_{\tau^-}^\pi - \beta \pi_\tau X_{\tau^-}^\pi \quad \text{on } \{\tau < \infty\}.$$

We make no assumption about the distribution of $(\tau, \beta)$. Instead, we assume that the investor is extremely risk averse with respect to a crash:

$$\sup_{\pi} \inf_{\tau, \beta} \mathbb{E} \left[ U_p \left( X_T^{\pi, \tau, \beta} \right) \right].$$
The stochastic optimization problem

$$\sup_{\pi} \inf_{\tau, \beta} \mathbb{E} \left[ U_p \left( X^{\pi, \tau, \beta}_T \right) \right]$$

can be understood as a game between the investor and some opponent.
The stochastic optimization problem

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The investor faces a balancing problem between a good expected rate of return and crash exposure.
Some Intuition for the Worst-Case Approach

The stochastic optimization problem

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The investor faces a balancing problem between a good expected rate of return and crash exposure.

**Literature:** Korn/Wilmott (2002), Korn/Menkens (2005), Korn/Steffensen (2007), Seifried (2010), Desmettre/Korn/Seifried (2014), ...
Solution of the Worst-Case Problem

Figure: The solution of the worst-case problem.
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This section is based on:


Our aim is to study the worst-case model in the presence of proportional transaction costs.
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We denote by $B$ the investor’s wealth in the bond and let $S$ denote the investor’s wealth in the stock. In the presence of proportional transaction costs, the wealth evolves as

\[
\begin{align*}
\mathrm{d} B_t &= r B_t \mathrm{d} t \\
\mathrm{d} S_t &= \alpha S_t \mathrm{d} t + \sigma S_t \mathrm{d} W_t
\end{align*}
\]

$B_0 = b, \quad S_0 = s$. 

• $L$ models the cumulative amount of money used for purchases of the stock.
• $M$ models the cumulative amount of money used for sales of the stock.
• $\lambda$ and $\mu$ model the costs for purchases and sales, respectively.

The investor’s wealth at the liquidation of the stock position is

\[
X_t = \begin{cases} 
B_t + (1 - \mu) S_t, & \text{if } S_t > 0 \\
B_t + (1 + \lambda) S_t, & \text{if } S_t \leq 0
\end{cases}
\]

We say that a trading strategy $(L,M)$ is admissible if $X_{L,M} \geq 0$. 

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The Financial Market Model

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\frac{dB_t}{B_t} &= \frac{r}{B_t} \, dt - (1 + \lambda) \, dL_t \\
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We say that a trading strategy $(L, M)$ is admissible if $X^{L,M} \geq 0$. 
Problem Formulation

In the absence of crashes, the optimization problem is given by

$$\mathcal{V}_0(t, b, s) \triangleq \sup_{\mathcal{O}_0 = (L, M)} \mathbb{E}_{t, b, s} \left[ U_p \left( X_{T}^{\mathcal{O}_0} \right) \right].$$
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In the presence of at most one crash, the optimization problem is given by

\[ \mathcal{V}_1(t, b, s) \triangleq \sup_{\mathcal{W}_1, \mathcal{W}_0} \inf_{\tau, \beta} \mathbb{E}_{t, b, s} \left[ U_p \left( X_T^{\mathcal{W}_1, \mathcal{W}_0, \tau, \beta} \right) \right]. \]

**Literature:** Magill/Constantinides (1976), Davis/Norman (1990), Shreve/Soner (1994), Akian/Séquier/Sulem (1995), Dai/Yi (2009), Herzog/Kunisch/Sass (2013), ...
Some Intuition on the Optimal Strategies

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- More precisely, we expect that the investor will try to keep her risky fraction in a neighborhood of $\pi_M$ and $\pi^*$, respectively, and only make small trades to keep the risky fraction from moving too far away.
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- More precisely, we expect that the investor will try to keep her risky fraction in a neighborhood of $\pi_M$ and $\pi^*$, respectively, and only make small trades to keep the risky fraction from moving too far away.
- There should hence be three regions:
  A no-trade region, a buy region and a sell region.
The Dynamic Programming Principle

The key tool in studying the stochastic optimization problems we face is the so-called dynamic programming principle (DPP):

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

— Richard Bellman, 1957
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**Theorem: Dynamic Programming [BMS 15]**

For every \([t, T]\)-valued stopping time \(\theta\), we have

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\mathcal{V}_0(t, b, s) = \sup_{\omega} \mathbb{E} \left[ \mathcal{V}_0(\theta, B_\theta^{\omega}, S_\theta^{\omega}) \right]
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\]

and

\[
\mathcal{V}_1(t, b, s) = \sup_{\omega_1} \inf_{\tau, \beta} \mathbb{E} \left[ \mathcal{V}_1(\theta, B_{\theta}^{\omega_1}, S_{\theta}^{\omega_1}) \mathbbm{1}_{\{\theta < \tau\}} \right. \\
+ \mathcal{V}_0(\tau, B_{\tau -}^{\omega_1}, (1 - \beta)S_{\tau -}^{\omega_1}) \mathbbm{1}_{\{\theta \geq \tau\}} \right].
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Arguing heuristically, the DPP can be used to derive partial differential equations for $V_0$ and $V_1$. These PDEs are the so-called dynamic programming equations.
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In the absence of crashes, we obtain the PDE

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\min \left\{ \mathcal{L}^{nt} V_0(t, b, s), \mathcal{L}^{buy} V_0(t, b, s), \mathcal{L}^{sell} V_0(t, b, s) \right\} = 0.
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In the presence of at most once crash, the PDE is

$$\max \left\{ V_1(t, b, s) - V_0(t, b, (1 - \beta^*) s), \right. \\
\left. \min \left\{ L^\text{nt} V_1(t, b, s), L^\text{buy} V_1(t, b, s), L^\text{sell} V_1(t, b, s) \right\} \right\} = 0.$$
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In the absence of crashes, we obtain the PDE

$$\min \left\{ \mathcal{L}^{nt} V_0(t, b, s), \mathcal{L}^{\text{buy}} V_0(t, b, s), \mathcal{L}^{\text{sell}} V_0(t, b, s) \right\} = 0.$$  

In the presence of at most once crash, the PDE is

$$\max \left\{ V_1(t, b, s) - V_0(t, b, (1 - \beta^*)s), \min \left\{ \mathcal{L}^{nt} V_1(t, b, s), \mathcal{L}^{\text{buy}} V_1(t, b, s), \mathcal{L}^{\text{sell}} V_1(t, b, s) \right\} \right\} = 0.$$  

Formally, these PDEs are second-order, parabolic free boundary problems.
Viscosity Solutions and Uniqueness

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Theorem: Viscosity Property of $V_0$ and $V_1$ \cite{BMS14,BMS15}

$V_0$ and $V_1$ are the unique viscosity solutions of their respective PDEs.

Remark: This approach enables us to study the PDEs numerically, but it does not establish existence of the optimal strategies.
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Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!
The Optimal Strategy in the Crash-Free Case

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There exists a strategy $\varpi^*$ which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.
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   c. ... defining $h^*(t, b, s) = \mathbb{E}_{t,b,s}[U_p(X^{\varpi^*}_T)]$ and showing that $h^* \in \mathbb{H}$. 

Conclusion:

• Our aim was to analyze a model which includes transaction costs and the possibility of severe market crashes.

• We proved several results: The dynamic programming principle, existence and uniqueness of viscosity solutions of the dynamic programming equations, construction of the candidate optimal strategy, verification of the optimality of the candidate optimal strategy, ...

• We extended the classical worst-case model to allow for a random/unbounded number of crashes and the possibility of financial bubbles.

• We solved this model completely, derived the optimal strategies and provided extensive numerical examples.

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The wealth dynamics are given as before

\[ dX_t = r(1 - \pi_t)X_t dt + \alpha \pi_t X_t dt + \sigma \pi_t X_t dW_t. \]
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\[
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\]

The switching between states is modeled through a continuous-time Markov chain $Z$ with finite state space $E = \{0, 1, \ldots, d\}$.

We allow for state-dependent market parameters and the investor can choose a different strategy $\pi^i$ for each state.
Schematic of the Model

Z in state 0

↔ No crash possible
Z in state 0

\( \rightarrow \) No crash possible

\( \downarrow \)

Z jumps to state \( i \)

\( \rightarrow \) Investor receives warning

\( \rightarrow \) Crash of maximum size \( \beta_i^* \geq 0 \) possible
Schematic of the Model

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\begin{align*}
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\leftarrow \\

\text{Crash } (\tau, \beta) \text{ occurs} \\
\rightarrow \text{ Stock price crashes by a fraction of } \beta \leq \beta_i^* \\
\rightarrow \text{ Z jumps back to state } 0
\end{align*}
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\( \leftrightarrow \)

\( Z \) jumps to state \( j \)

\( \leftrightarrow \) Investor receives new information

\( \leftrightarrow \) Crash of maximum size \( \beta^*_j \geq 0 \) possible
Figure: Optimal strategies in the case of a random number of crashes.
Optimal Strategies: Example II

Figure: Optimal strategies in the case of a random number of crashes.
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The Dynamic Programming Equation for $\beta_i^* = 0$

Denote by $(q_{i,j})_{0 \leq i,j \leq d}$ the generator matrix of $Z_t$ and let

$$L^\pi_i V \triangleq \frac{\partial}{\partial t} V + \alpha_i \pi x \frac{\partial}{\partial x} V + \frac{1}{2} \sigma_i^2 \pi^2 x^2 \frac{\partial^2}{\partial x^2} V, \quad i = 0, \ldots, d.$$
The Dynamic Programming Equation for $\beta^*_i > 0$

Define the following sets:

- $A_1 \triangleq \left\{ \pi \in \mathcal{K} : \mathcal{V}(t, x, i) \leq \mathcal{V}(t, (1 - \beta^*_i \pi)x, 0) \right\}$,

- $A_2 \triangleq \left\{ \pi \in \mathcal{K} : \mathcal{L}^\pi_i \mathcal{V}(t, x, i) + \sum_{j=1}^{d} q_{i,j} \mathcal{V}(t, x, j) \geq 0 \right\}$.

The value function $\mathcal{V}(\cdot, \cdot, i)$ and the corresponding optimal strategy $\pi^*_{\cdot, \cdot, i}$ in state $i$ can be determined by solving

\[
0 \leq \sup_{\pi \in A_1} \left[ \mathcal{L}^\pi_i \mathcal{V}(t, x, i) + \sum_{j=1}^{d} q_{i,j} \mathcal{V}(t, x, j) \right],
\]

\[
0 \leq \sup_{\pi \in A_2} \left[ \mathcal{V}(t, (1 - \beta^*_i \pi)x, 0) - \mathcal{V}(t, x, i) \right],
\]

\[
0 = \sup_{\pi \in A_1} \left[ \mathcal{L}^\pi_i \mathcal{V}(t, x, i) + \sum_{j=1}^{d} q_{i,j} \mathcal{V}(t, x, j) \right] \cdot \sup_{\pi \in A_2} \left[ \mathcal{V}(t, (1 - \beta^*_i \pi)x, 0) - \mathcal{V}(t, x, i) \right].
\]
The optimal strategy in state $i$ with $\beta_i^* > 0$ is given by $\pi_t^{i,*} \triangleq \min\{\pi_t^i, \pi_t^{i,\text{ind}}\}$, where $\pi_t^{i,\text{ind}}$ solves the differential equation

$$\frac{\partial}{\partial t} \pi_t^{i,\text{ind}} = \frac{1}{\beta_i^*} (1 - \pi_t^{i,\text{ind}} \beta_i^*) \left[ \Psi_i - \Psi_0 - \frac{1}{2} (1 - p) \sigma_i^2 \left( \pi_t^{i,\text{ind}} - \pi_M^i \right)^2 ight. $$

$$+ \frac{1}{p} \sum_{j=0}^{d} q_{i,j} \frac{f_j(t)}{f_0(t)} (1 - \pi_t^{i,\text{ind}} \beta_i^*)^{-p} $$

$$- \frac{1}{p} \sum_{j=0 \atop j \neq i}^{d} q_{0,j} \frac{f_j(t)}{f_0(t)} - q_{0,i} \frac{1}{p} (1 - \pi_t^{i,\text{ind}} \beta_i^*)^p \right].$$

Here, $\Psi_i \triangleq \frac{1}{2} \frac{\alpha_i^2}{(1-p)\sigma_i^2}$, and the functions $f_i$ solve the system of ODEs

$$\frac{\partial}{\partial t} f_i(t) = -p \alpha_i \min \left\{ \frac{1}{\beta_i^*} \left(1 - \left[ \frac{f_i(t)}{f_0(t)} \right]^{1/p} \right), \pi_M^i \right\} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t) $$

$$+ \frac{1}{2} p(1 - p) \sigma_i^2 \left[ \min \left\{ \frac{1}{\beta_i^*} \left(1 - \left[ \frac{f_i(t)}{f_0(t)} \right]^{1/p} \right), \pi_M^i \right\} \right]^2 f_i(t).$$