

Backward Nonlinear Expectation Equations

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Joint work with **Frank Seifried** and **Thomas Seiferling**

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BNEEs can be regarded as the equivalent of backward stochastic differential equations in a nonlinear expectation setting.

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- (6) $\mathcal{E}_t[X + Y] \leq \mathcal{E}_t[X] + \mathcal{E}_t[Y]$. (sub-additivity)

Appropriate Family of L^p spaces

Let Ω be a non-empty set. We take as given a family

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In the sequel we write $L^p \triangleq L_T^p$ for simplicity.

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Existence and Uniqueness of BNEEs

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Let \mathcal{E}_t be a **measurable** nonlinear expectation and assume that

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as well as the discrete-time **backward aggregation** $X_N^N = \xi$,

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for a partition $\Delta^N : 0 = t_1 < \dots < t_N = T$, $\Delta_k^N \triangleq t_k - t_{k-1}$. Assume that $(\Delta^N)_{N \in \mathbb{N}}$ is **refining** and **infinitesimal**.

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Convergence Theorem

If $t \mapsto g(t, Y)$ is càdlàg for every $Y \in L^p$, then

$$\max_{k=0, \dots, N} \|X_k^N - X_{t_k}\|_{L^p} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

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