Worst-Case Portfolio Optimization

Bubbles and Transaction Costs

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Based on joint work with **Sören Christensen**, **Olaf Menkens** and **Jörn Sass**.

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Overview

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- ... we want to make a long-term investment,
- ... we want to maximize our wealth at the end of the investment period,
- ... the market may possibly crash.

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The function U_p is called the **utility function** and models the investor's **risk preferences**. We assume that

$$U_p(x) = \begin{cases} \frac{1}{p}x^p, & \text{if } p < 1, p \neq 0, \\ \log(x), & \text{if } p = 0. \end{cases}$$

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2. We should expect the process $\mathcal{V}_0(u, X^{\pi^*}_{t,x}(u))$ to be an **honest martingale** for the optimal π^*.

The Hamilton-Jacobi-Bellman Equation

The dynamic programming principle leads to the following PDE – the **Hamilton-Jacobi-Bellman** (HJB) equation:

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{V}_0 + (r + (\alpha - r)\pi)x\frac{\partial}{\partial x}\mathcal{V}_0 + \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2}{\partial x^2}\mathcal{V}_0 &\leq 0, \\ \mathcal{V}_0(T, x) &= U_p(x). \end{aligned}$$

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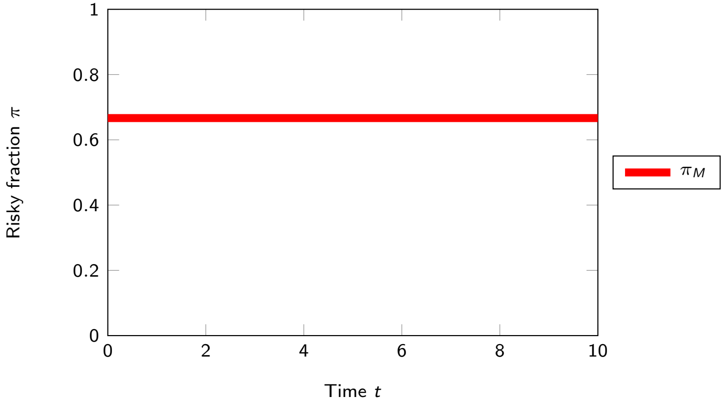
This equation is solved explicitly by

$$\mathcal{V}_0(t,x) = \frac{1}{p}x^p\exp\left\{p\left(r-\frac{1}{2(1-p)}\frac{(\alpha-r)^2}{\sigma^2}\right)(T-t)\right\}$$

and the **optimal strategy** is given by

$$\pi_0^*(t) = \pi_M := \frac{\alpha-r}{(1-p)\sigma^2}.$$

The Solution to the Merton Problem



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Observation: If a crash occurs close to the end of the investment period the investor will suffer substantial losses.

Overview

The Worst-Case Approach of Korn/Wilmott (2002)

We assume that a crash is given by a pair $\vartheta = (\tau, \bar{\beta})$. At the stopping time τ, the **stock price crashes** by the fraction $\bar{\beta}$:

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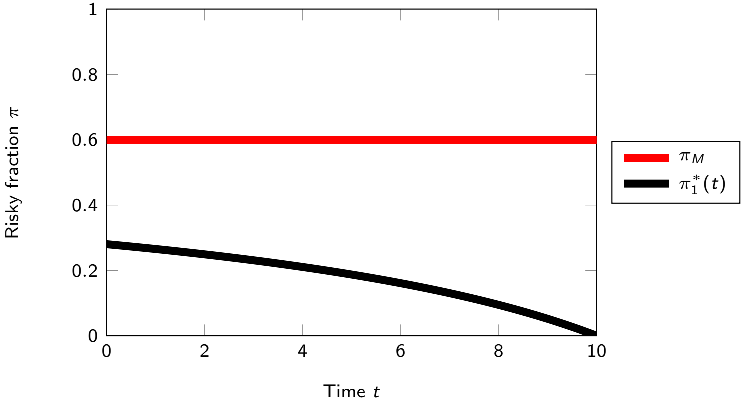
Moreover, it is straightforward to verify that **this strategy is optimal**!

The Optimal Strategy in the Worst-Case Model



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2. extend the model to allow for **transaction costs**.

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- We allow for **state-dependent market parameters**.

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Whenever $\beta_i = 0$, the investor **does not have to fear a crash**. Following the solution approach as in the Merton model, we expect that $\mathcal{V}(t, x, i)$ solves an HJB equation of a similar form.

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The **value function** $\mathcal{V}(\cdot_1, \cdot_2, i)$ and the corresponding **optimal strategy** π_i^* in state i with $\beta_i = 0$ can be determined by solving the following HJB equation:

$$0 \geq \qquad \mathcal{L}_i^{\pi} \mathcal{V}(t, x, i) + \sum_{j=0}^{d} q_{i,j} \mathcal{V}(t, x, j) \ .$$

The HJB Equation for States with $\beta_i = 0$

Whenever $\beta_i = 0$, the investor **does not have to fear a crash**. Following the solution approach as in the Merton model, we expect that $\mathcal{V}(t, x, i)$ solves an HJB equation of a similar form.

Denote by $(q_{i,j})_{0 \le i,j \le d}$ the **generator** matrix of Z and let

$$\mathcal{L}_i^\pi \mathcal{V} = \frac{\partial}{\partial t}\mathcal{V} + (r_i + (\alpha_i - r_i)\pi)x\frac{\partial}{\partial x}\mathcal{V} + \frac{1}{2}\sigma_i^2\pi^2x^2\frac{\partial^2}{\partial x^2}\mathcal{V}, \quad i = 0, \dots, d.$$

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The HJB Equation for States with $\beta_i > 0$

$$\mathcal{V}(t, x, i)$$

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$$\mathcal{V}(t, x, i) \quad \mathcal{V}(t, (1 - \pi\beta_i)x, 0)$$

The HJB Equation for States with $\beta_i > 0$

$$\mathcal{V}(t, x, i) \leq \mathcal{V}(t, (1 - \pi\beta_i)x, 0)$$

The HJB Equation for States with $\beta_i > 0$

Define the following sets:

$$A_1 := \Big\{\pi : \mathcal{V}(t,x,i) \le \mathcal{V}(t,(1-\pi\beta_i)x,0)\Big\},$$

The value function $\mathcal{V}(\cdot,\cdot,i)$ and the corresponding optimal strategy π_i^* in state i with β_i can be determined by solving the **HJB system**

$$0 \le \sup_{\pi \in A_1} \Big[\mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j}\mathcal{V}(t,x,j)\Big],$$

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$$0 \leq \sup_{\pi\in A_1}\Big[\mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j}\mathcal{V}(t,x,j)\Big],$$

$$0 \leq \sup_{\pi\in A_2}\Big[\mathcal{V}(t,(1-\pi\beta_i)x,0) - \mathcal{V}(t,x,i)\Big],$$

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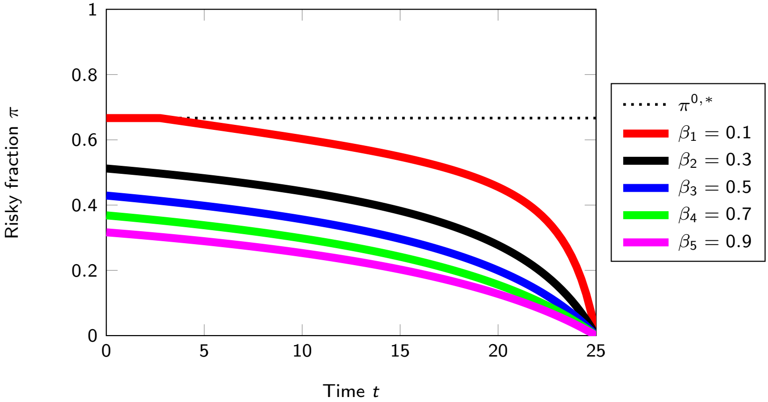
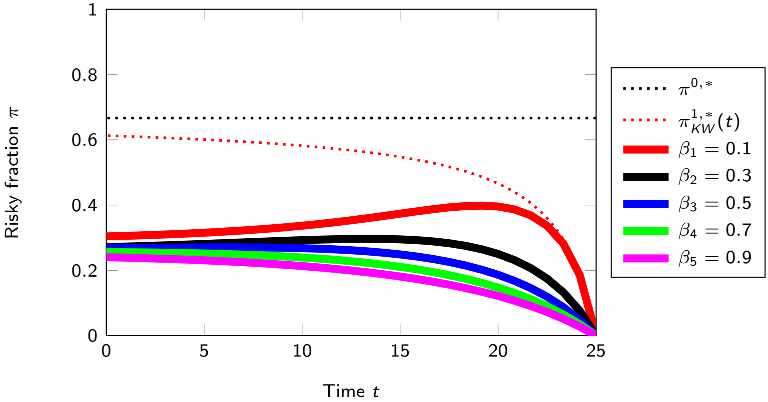
$$0 = \sup_{\pi\in A_1}\Big[\mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j}\mathcal{V}(t,x,j)\Big] \cdot \sup_{\pi\in A_2}\Big[\mathcal{V}(t,(1-\pi\beta_i)x,0) - \mathcal{V}(t,x,i)\Big].$$

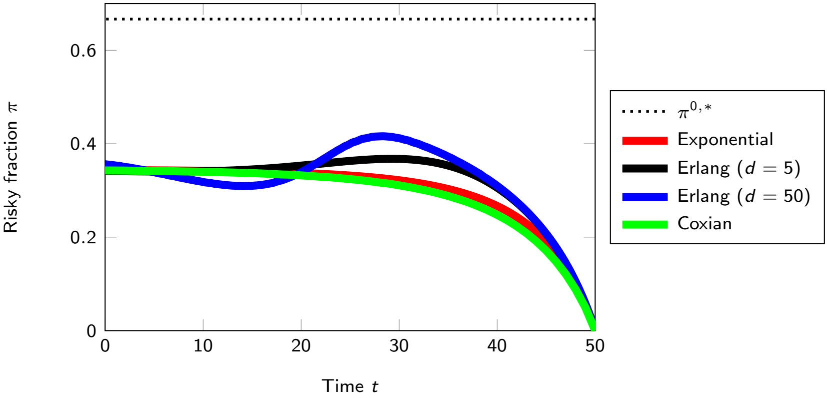
An Example


1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
Risky fraction π
Time t
$\pi^{0,*}$
$\beta_1 = 0.1$
$\beta_2 = 0.3$
$\beta_3 = 0.5$
$\beta_4 = 0.7$
$\beta_5 = 0.9$

Another Example



One more...


0.6
0.4
0.2
0
0
10
20
30
40
50
Risky fraction π
Time t
π^{0,*}
Exponential
Erlang (d = 5)
Erlang (d = 50)
Coxian

Overview

Trading under Proportonial Transaction Costs

Let us now consider the Korn/Wilmott model (one crash only) in the presence of **transaction costs**, i.e. we assume that the investor buys and sells shares of the stock at the price

$$(1+\lambda)P_1(t) \quad \text{and} \quad (1-\mu)P_1(t),$$

respectively, where $\lambda > 0$, $\mu \in (0,1)$.

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Trading strategies are modeled as a pair (L, M). The **wealth** invested in the bond and stock is assumed to be given by

$$\begin{aligned} dB(t) &= rB(t)dt & B(0) &= b, \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) & S(0) &= s. \end{aligned}$$

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Problem Formulation

The investor's **net wealth** X in this model is given by

$$X(t) = \begin{cases} B(t) + (1-\mu)S(t), & \text{if } S(t) > 0, \\ B(t) + (1+\lambda)S(t), & \text{if } S(t) \leq 0. \end{cases}$$

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The optimization problems in this setting are given by

$$\mathcal{V}_0(t,b,s) = \sup_{(L,M)} \mathbb{E}\Big[U_p\Big(X^{L,M}_{t,b,s}(T)\Big)\Big]$$

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$$\mathcal{V}_1(t,b,s) = \sup_{\substack{\varpi_1=(L_1,M_1) \\ \varpi_0=(L_0,M_0)}} \inf_{\tau \in \mathcal{B}(t)} \mathbb{E}\Big[U_p\Big(X^{\varpi_1,\varpi_0,\tau}_{t,b,s}(T) \Big) \Big]$$

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- More precisely, we expect that the investor will try to keep her risky fraction **in a neighborhood of the Merton fraction** and only makes small trades to keep the risky fraction from moving too far away.
- There should hence be three regions: A **no-trading region**, a **buying region** and a **selling region**.

The HJB Equation in the Absence of Crashes

The HJB equation in the **absence of crashes** takes the following form

$$0 = \min\left\{\mathcal{L}^{nt}\mathcal{V}_0(t,b,s), \mathcal{L}^{buy}\mathcal{V}_0(t,b,s), \mathcal{L}^{sell}\mathcal{V}_0(t,b,s)\right\},$$

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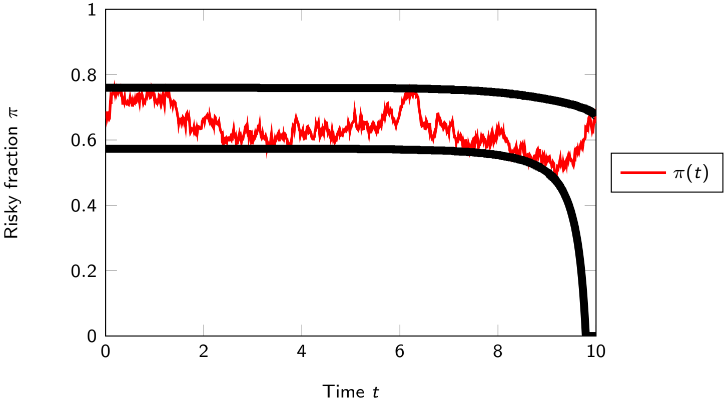
But: We can show that the the value function is a weak (viscosity) solution of the HJB equation and determine the trading regions numerically.

A Numerical Example



The HJB Equation in the Presence of Crashes

In the **presence of crashes**, the HJB equation is given by

$$0 = \max\Big\{ \mathcal{V}_1(t,b,s) - \mathcal{V}_0(t,b,(1-\beta)s), \\ \min\left\{ \mathcal{L}^{nt}\mathcal{V}_1(t,b,s), \mathcal{L}^{buy}\mathcal{V}_1(t,b,s), \mathcal{L}^{sell}\mathcal{V}_1(t,b,s) \right\} \Big\}.$$

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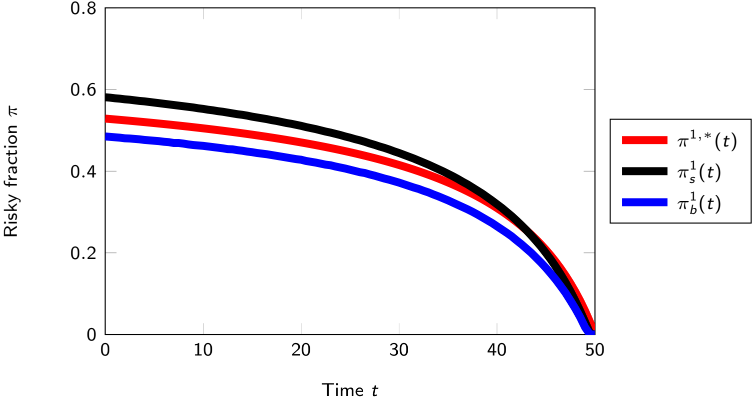
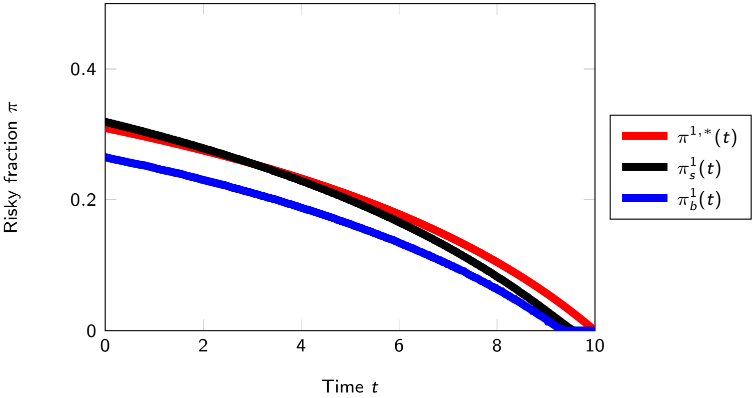
As in the case without crashes, we can show that $\mathcal{V}_1$ solves this equation in the weak (viscosity) sense.

Trading Regions in the Presence of Crashes: Long term



Trading Regions in the Presence of Crashes: Short term



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- The assumption of a fixed number of crashes can be relaxed and we can even model **financial bubbles**.
- In the presence of **transaction costs**, the investor keeps her risky fraction in a region around the optimal no-cost strategy.
- In the presence of crashes, it may sometimes **not be optimal** to invest any money in the stock at all.

Literature

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