Recursive Utility and Stochastic Differential Utility

with Nonlinear Expectations

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TUK-TUM Wanderseminar in Kaiserslautern
October 10, 2014

Overview

Motivation: Comparison of Consumption Plans

Comparing consumption plans

Let $\mathcal{C}$ be a set of consumption plans (c, X) and suppose you have to decide today which of the plans you want to follow.

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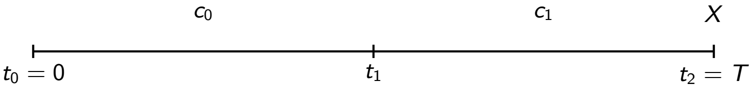
Example: Discounted expected utility:

$$\mathfrak{v}(c, X) = \mathbb{E}\left[\sum_{n=0}^{N-1} e^{-\delta t_n} U(c_{t_n}) + e^{-\delta T} U(X)\right],$$

$$\mathfrak{v}(c, X) = \mathbb{E}\left[\int_0^T e^{-\delta t} U(c_t)\, dt + e^{-\delta T} U(X)\right].$$

Recursive Utility (Kreps/Porteus 1978): Discrete Time



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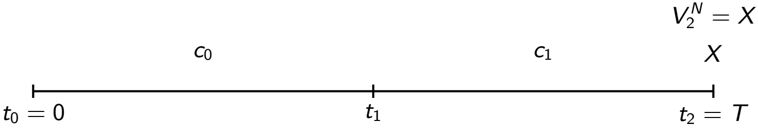

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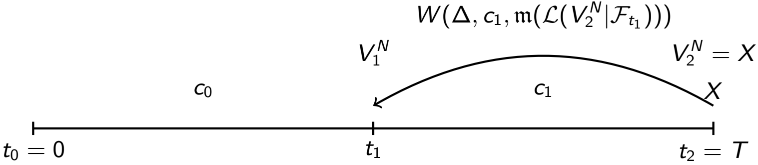

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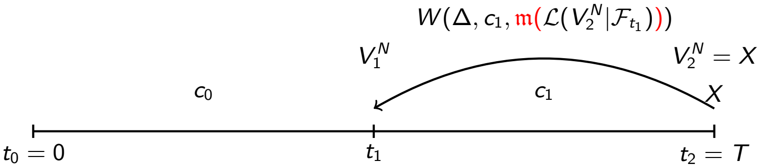

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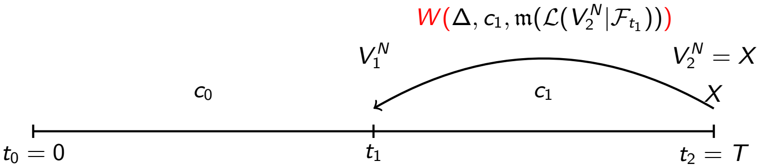

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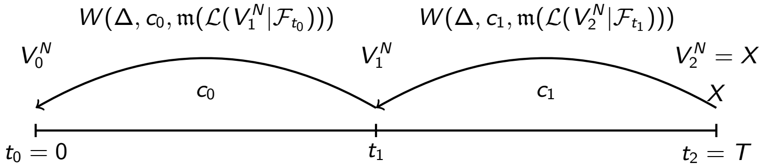

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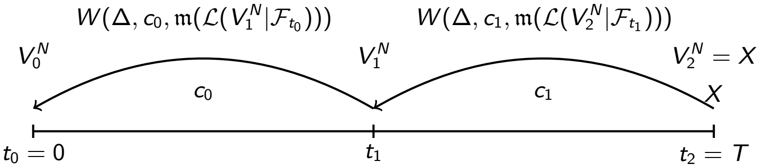

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$$\overset{\Delta=1}{\Rightarrow} \mathfrak{v}(c, X) = V_0^N = \ldots = U^{-1}\left(\mathbb{E}\left[\sum_{i=0}^{1} e^{-\delta t_i}U(c_i) + e^{-\delta T}U(X)\right]\right).$$

Stochastic Differential Utility (Duffie/Epstein 1992): Continuous Time

In a continuous-time setting, **stochastic differential utility** is defined axiomatically through

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where $V = (V_t)_{t \in [0,T]}$ is given as the solution of the BSDE

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Example: If $f(c, v) = U(c) - \delta v$, then

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In particular,

$$f(c, v) = \frac{\partial}{\partial \Delta} W(\Delta, c, v)|_{\Delta=0}.$$

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We say that $\mathcal{E}_t$ is a **sublinear expectation** if additionally

$$\mathcal{E}_t[X + Y] \leq \mathcal{E}_t[X] + \mathcal{E}_t[Y].$$

Extension of Sublinear Expectations

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{F}_t$ be a filtration such that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. The set $\mathcal{H}$ is assumed to be a linear space of bounded $\mathcal{F}$-measurable random variables such that $1 \in \mathcal{H}$ and $|X|, XY \in \mathcal{H}$ whenever $X, Y \in \mathcal{H}$. Moreover, $\mathcal{H}_t \subset \mathcal{H}$ denotes the subset of all $\mathcal{F}_t$-measurable random variables.

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Let $\mathcal{E}_t^{sub}$ be a **sublinear expectation**. We want to define a seminorm on $\mathcal{H}$ by

$$\|X\|_{L,p}^p := \mathcal{E}_0^{sub}[|X|^p]$$

and **extend** $\mathcal{E}_t^{sub}$ from $\mathcal{H}$ to its completion L^p under $\|\cdot\|_{L,p}$.

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Problem: This is an **abstract completion**.

Representation of Sublinear Expectations

The following **representation** can be used to extend $\mathcal{E}_0^{sub}$ to all measurable X with $\sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[X^-] < \infty$.

Representation by finitely additive probability measures

$\mathcal{E}_0^{sub}$ can be represented as

$$\mathcal{E}_0^{sub}[X] = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X], \qquad X \in \mathcal{H},$$

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If $(\mathcal{E}_0^{sub}, \mathcal{Q})$ satisfies

$$\mathcal{E}_0^{sub}[\liminf_{n\to\infty} X_n] \leq \liminf_{n\to\infty} \mathcal{E}_0^{sub}[X_n]$$

for every sequence of non-negative random variables $(X_n)_{n\in\mathbb{N}}$, then we can **extend** $(\mathcal{E}_0^{sub}, \mathcal{Q})$ to a Banach space $L^p_{\mathcal{Q}}$ and **identify** L^p with a closed subspace.

Nonlinear Expectations and Processes

In a similar fashion, we can extend a nonlinear expectation $\mathcal{E}_t$ to L^p provided that it is **dominated** by $\mathcal{E}_t^{sub}$ in the following sense:

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We can moreover construct **Banach spaces** P^p and S^p of **processes** $(X_t)_{t\in[0,T]}$ through the seminorms

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Naturally: $X \in L^p$, $c \in P^p$, $c^N \in P^p_{step}$, $V^N, V \in S^p$.

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3. $|\varepsilon(\Delta, c, v)| \leq h(\Delta)(1 + |c| + |v|)$ with $h(0) = 0$.

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We have

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1. No **linearity**!
2. We have to ensure that everything is **well-defined**.
3. While $\mathcal{E}^{sub}$ satisfies Fatou's lemma it generally does not satisfy **dominated convergence**.

Thank you for your attention!!!