Worst-Case Portfolio Optimization

in a Market with Bubbles

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Overview

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The Worst-Case Problem

$$\sup_{\pi} \inf_{\theta} \mathbb{E}\left[U(X_T^{\pi,\theta})\right].$$

Literature

Literature related to **Bubbles**:

Loewenstein and Willard (2000), Cox and Hobson (2005), Jarrow, Protter and Shimbo (2007, 2010), Biagini, Föllmer and Nedelcu (2013), ...

$\Rightarrow$ Compare with Föllmer's talk.

Literature related to the **Worst-Case Approach**:

Korn and Willmot (2002), Korn and Menkens (2005), Korn and Steffensen (2007), Seifried (2010), ...

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The Market

We start with a **Black-Scholes market**:

$$dB_t = 0, \qquad dS_t = \alpha S_t dt + \sigma S_t dW_t.$$

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- After a crash, Z_t is assumed to **jump back to state 0**.

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$\hookrightarrow$ No crash possible

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We say that a strategy $\pi = (\pi^0, \dots, \pi^d)$ is **admissible**, if it leads to nonnegative wealth **for all possible crash scenarios** $\theta = (\tau_k, \kappa_k)_{k \in \mathbb{N}}$. This holds if

$$\pi^i \le \frac{1}{\kappa^i}, \qquad \text{for all } i.$$

The Worst-Case Problem

The aim of the investor is to find the strategy $\pi^* = (\pi^{0,*}, \dots, \pi^{d,*})$ which **performs best** if the **worst-possible crash scenario** $\theta = (\tau_k, \kappa_k)_{k \in \mathbb{N}}$ occurs.

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Here, U is assumed to be the **power utility** function:

$$U(x) = \frac{1}{\gamma} x^{\gamma}, \qquad \gamma < 1, \gamma \neq 0.$$

Overview

1. Introduction and Model Setup
2. The HJB System and Verification
3. Numerical Results

The HJB Equation for $i = 0$

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The value function $V(\cdot, \cdot, 0)$ and the corresponding optimal strategy in state 0 can be determined by solving the following **HJB equation**:

$$0 = \sup_{\pi} \left[\mathcal{L}^{\pi} V(t, x, 0) + \sum_{j=0}^{d} q_{0,j} V(t, x, j) \right].$$

The HJB Equation for $i = 0$

Denote by $(q_{i,j})_{0 \le i,j \le d}$ the generator matrix of Z_t and let

$$\mathcal{L}^{\pi} V = V_t + \alpha \pi x V_x + \frac{1}{2} \sigma^2 \pi^2 x^2 V_{xx}.$$

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Verification for $i = 0$

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The **value function** is given by

$$V(t,x,0) = \frac{1}{\gamma} x^{\gamma} f^0(t),$$

where $f^0(t)$ solves

$$f_t^0(t) = -\frac{\gamma\alpha^2}{2(1-\gamma)\sigma^2} f^0(t) - \sum_{j=0}^{d} q_{0,j} f^j(t), \qquad f^0(T) = 1.$$

The HJB Equation for $i > 0$

Define the following sets:

$$A_1 := \left\{ \pi : V(t,x,i) \leq V(t,(1-\kappa^i \pi_t)x,0) \right\},$$

The value function $V(\cdot,\cdot,i)$ and the corresponding optimal strategy in state i can be determined by solving the **HJB system**

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$$0 = \sup_{\pi\in A_1}\Big[\mathcal{L}^\pi V(t,x,i) + \sum_{j=1}^{d} q_{i,j} V(t,x,j)\Big] \cdot \sup_{\pi\in A_2}\Big[V(t,(1-\kappa^i\pi_t)x,0) - V(t,x,i)\Big].$$

Verification for $i > 0$

Suppose **both equations** are equal to 0 at the same time. As before, we make the ansatz

$$V(t,x,i) = \frac{1}{\gamma} x^{\gamma} f^{i}(t).$$

With this, the equations reduce to

$$f^{i}(t) = (1 - \pi_t^{i,*}\kappa^{i})^{\gamma} f^{0}(t),$$

$$\frac{\partial}{\partial t} f^{i}(t) = -\gamma f^{i}(t)\left(\alpha\pi_t^{i,*} - \frac{1}{2}(1-\gamma)\sigma^2(\pi_t^{i,*})^2\right) - \sum_{j=1}^{d} q_{i,j} f^{j}(t).$$

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Taking the logarithm in the first equation and then the derivative with respect to t, this yields

$$\frac{\partial}{\partial t} \pi_t^{i,*} = \frac{1}{\gamma \kappa^i} (1 - \pi_t^{i,*} \kappa^i) \left[\frac{1}{f^0(t)} \frac{\partial}{\partial t} f^0(t) - \frac{1}{f^i(t)} \frac{\partial}{\partial t} f^i(t) \right].$$

Verification for $i > 0$

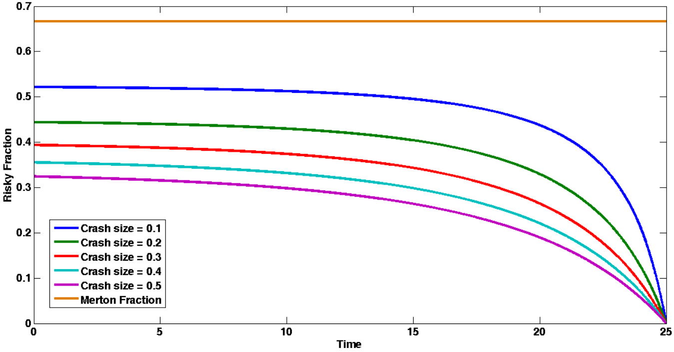
Plugging the ODEs for f^0 and f^i into the last equation, we arrive at

$$\frac{\partial}{\partial t}\pi_t^{i,*} = -\frac{1}{\kappa^i}(1-\pi_t^{i,*}\kappa^i)\Bigg[\frac{1}{2}(1-\gamma)\sigma^2\big(\pi_t^{i,*}-\pi_t^{0,*}\big)^2 - \frac{1}{\gamma}\sum_{j=1}^{d} q_{0,j}\Big((1-\pi^{j,*}\kappa^j)^\gamma - 1\Big) + \frac{1}{\gamma}\sum_{j=1}^{d} q_{i,j}\frac{(1-\pi_t^{j,*}\kappa^j)^\gamma}{(1-\pi_t^{i,*}\kappa^i)^\gamma}\Bigg],$$

with terminal condition $\pi_T^{i,*} = 0$.

Overview

Numerical Example



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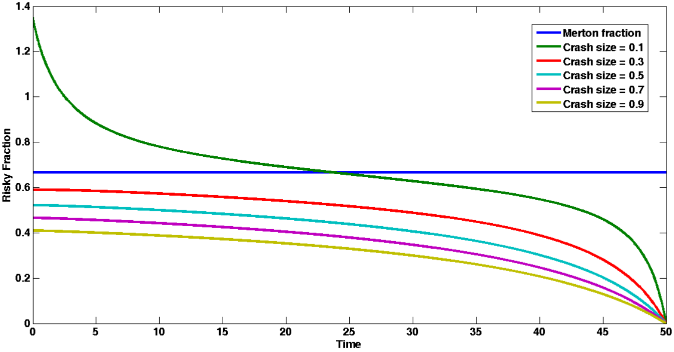

Conclusions

We propose a **regime switching model** for an optimal investment problem in a financial market with **bubbles**.

We derive a **system of HJB equations** which lead to a **coupled system of ordinary differential equations** for the optimal strategies.

Depending on the choice of parameters, the optimal strategies **may or may not make the investor indifferent** towards the impact of the crashes.

It is straightforward to extend the results to **state-dependent market coefficients**.

Thank you for your attention!!!