Worst-Case Portfolio Optimization in a Market with Bubbles

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Abstract

We study a portfolio optimization problem in a financial market which is under the threat of crashes in the risky asset. We assume that at random times, the investor receives warnings that a bubble has formed in the market which may lead to a crash. We make no assumptions about the distribution of this crash, but assume that the investor takes a worst-case perspective towards its impact.

The Market Model

We assume that the investor has access to two assets, a risk-free bond $(B_t)_{t\geq 0}$ with price evolution

$$dB_t = 0,$$

and a risky stock $(S_t)_{t\geq 0}$ with price dynamics

$$dS_t = \alpha S_t dt + \sigma S_t dW_t.$$

Here, $\alpha > 0$ and $\sigma > 0$ are assumed to be constant and $(W_t)_{t\geq 0}$ denotes a standard Brownian motion.

Warnings and Crashes

Let $(Z_t)_{t\geq 0}$ be a continuous-time Markov process with finite state space $\{0, \dots, d\}$. Each of the states $1, \dots d$ corresponds to a market regime in which one crash of maximum relative size $0 < \kappa^1, \dots, \kappa^d < 1$ may occur. After a crash has occurred, we assume that Z_t jumps to state 0, in which crashes are not possible.

A crash is modeled as a pair (τ, κ), where τ is a stopping time and κ denotes the relative size of the crash. That is, if the crash (τ, κ) occurs, then at time τ the stock price drops by the fraction κ:

$$S_\tau = (1-\kappa)S_{\tau-}.$$

Z_t in state 0
↪ No crash possible

↓

Z_t jumps to state i
↪ Investor receives warning
↪ Crash of maximum size κ^i possible

↙ ↘

Crash (τ, κ) occurs
↪ Stock price crashes by a fraction of κ
↪ Z_t jumps back to state 0

Z_t jumps to state j
↪ Investor receives new information
↪ Crash of maximum size κ^j possible

Trading Strategies and Wealth

For each state $i = 0, \dots, d$, the investor is allowed to choose which fraction $(\pi_t^i)_{t\geq 0}$ of her total wealth to invest into the risky asset. With this, the investor's wealth $(X_t)_{t\geq 0}$ evolves as

$$dX_t = \alpha\pi_t^i X_t dt + \sigma\pi_t^i X_t dW_t, \qquad \text{on } \{Z_t = i\},$$

and if a crash (τ, κ) occurs, then the wealth satisfies

$$X_\tau = (1 - \pi_\tau^i \kappa)X_{\tau-}.$$

Problem Formulation

We denote by $\mathcal{B}$ the set of sequences of crashes $(\tau_k, \kappa_k)_{k\in\mathbb{N}}$ such that in between two jump times of Z_t at most one crash occurs. We let furthermore $\mathcal{A}(t,x)$ be the set of all trading strategies such that for an initial wealth of $x > 0$ at time t, the wealth process is nonnegative for all crash scenarios $(\tau_k, \kappa_k)_{k\in\mathbb{N}} \in \mathcal{B}$. The investor's aim is to maximize

$$V(t,x,i) := \sup_\pi \inf_{(\tau_k,\kappa_k)_{k\in\mathbb{N}}} \mathbb{E}_{(t,x,i)}\big[U(X_T)\big],$$

where $\mathbb{E}_{(t,x,i)}$ denotes the conditional expectation given that $X_t = x$ and $Z_t = i$. The utility function $U(\cdot)$ is assumed to be either power or log utility:

$$U(x) = \begin{cases} \frac{1}{p}x^p, & \text{if } p < 1, p \neq 0, \\ \log x, & \text{if } p = 0. \end{cases}$$

Direct Verification for $d = 1$

In the special case of only one crash regime (i.e. $d = 1$), one can construct the optimal strategies directly. First, note that in this case the jump times of Z_t are simply exponential times with some parameter $\lambda > 0$. It is not difficult to see that in state 0 (no crash threat), the optimal strategy $\pi_t^{0,*}$ is the Merton strategy given by

$$\pi_t^{0,*} = \frac{\alpha}{(1-p)\sigma^2}.$$

For state 1, it is possible to construct a strategy $\pi_t^{1,*}$ which renders the investor indifferent between an immediate crash and no crash at all. Heuristically, one starts with

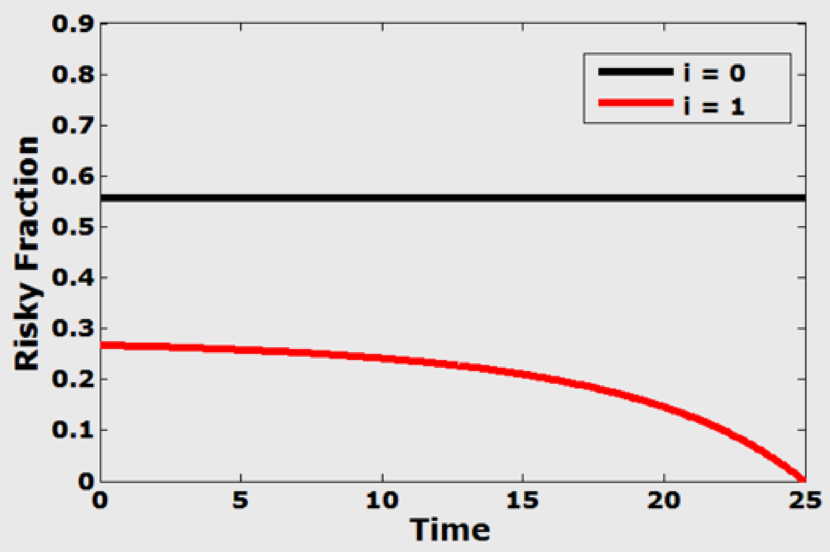
$$V(t,x,1) = V\big(t, (1-\pi_t^{1,*}\kappa^1)x, 0\big)$$

to derive the following differential equation for $\pi_t^{1,*}$ (in the power utility case):

$$\frac{\partial}{\partial t}\pi_t^{1,*} = -\frac{1}{\kappa^1}(1-\pi_t^{1,*}\kappa^1)\Big[\frac{1}{2}(1-p)\sigma^2\big(\pi_t^{1,*} - \pi_t^{0,*}\big)^2 + \frac{\lambda}{p}\big((1-\pi_t^{1,*}\kappa^1)^p - 1\big)\Big]$$

with $\pi_T^{1,*} = 0$. One can then verify directly that this strategy outperforms every other admissible strategy.

Numerical Example ($d = 1$)



$\alpha = 0.05, \sigma = 0.3, p = 0, \kappa^1 = 0.3, T = 25, \lambda = 1/T.$

The HJB System

Denote by $(q_{i,j})_{0\leq i,j\leq d}$ the generator of Z_t and let

$$\mathcal{L}^\pi = \frac{\partial}{\partial t} + \alpha\pi x\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2}{\partial x^2}.$$

One can show that $V(t,x,0)$ solves

$$0 = \sup_\pi\Big[\mathcal{L}^\pi V(t,x,0) + \sum_{j=0}^d q_{0,j}V(t,x,j)\Big]$$

and that the optimal strategy $\pi_t^{0,*}$ in state 0 is again the Merton strategy. For $i = 1, \dots, d$, the value function $V(t,x,i)$ is given as the solution of

$$0 = \min\Big\{\sup_\pi\Big[\mathcal{L}^\pi V(t,x,i) + \sum_{j=0}^d q_{i,j}V(t,x,j)\Big], \sup_\pi\Big[V\big(t,(1-\pi\kappa^i)x,0\big) - V(t,x,i)\Big]\Big\}.$$

In this case, the optimal strategy $\pi_t^{i,*}$ is the solution of

$$\frac{\partial}{\partial t}\pi_t^{i,*} = -\frac{1}{\kappa^i}(1-\pi_t^{i,*}\kappa^i)\Big[\frac{1}{2}(1-p)\sigma^2\big(\pi_t^{i,*} - \pi_t^{0,*}\big)^2 + \frac{1}{p}\Big(q_{0,0} + \sum_{j=1}^d q_{0,j}(1-\pi_t^{j,*}\kappa^j)^p\Big) - \frac{1}{p}\sum_{j=1}^d q_{i,j}\frac{(1-\pi_t^{j,*}\kappa^j)^p}{(1-\pi_t^{i,*}\kappa^i)^p}\Big],$$

with terminal condition given by $\pi_T^{i,*} = 0$.

Extensions and Outlook

The HJB System approach is flexible enough to extend the results to market coefficients which depend on the Markov process Z_t.

The model described here allows for possibly infinitely many crashes. It is possible to obtain similar results if we limit the maximum number of crashes to some constant n. As $n \to \infty$, one can show that the corresponding value functions and optimal strategies converge.

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