Uniqueness of Unbounded Viscosity Solutions

arising in Portfolio Optimization with Proportional Transaction Costs

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The Portfolio Problem

Optimal Terminal Wealth Problem

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Literature: Akian et al. (1995), Dai/Yi (2009), Bichuch (2012).

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Utility: Constant Relative Risk Aversion

$$U(x) = \frac{1}{p}x^p, \qquad p < 1, p \neq 0.$$

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The Hamilton-Jacobi-Bellman Equation

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$$0 = \min\left\{\mathcal{L}^{nt} V(t,b,s),\ \mathcal{L}^{buy} V(t,b,s),\ \mathcal{L}^{sell} V(t,b,s)\right\}$$

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Differential Operators:

$$\begin{aligned}\mathcal{L}^{nt}V &= -V_t - rbV_b - \alpha s V_s - \frac{1}{2}\sigma^2 s^2 V_{ss},\\ \mathcal{L}^{buy}V &= (1+\lambda)V_b - V_s,\\ \mathcal{L}^{sell}V &= -(1-\mu)V_b + V_s.\end{aligned}$$

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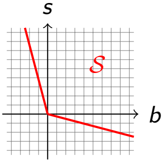
Domain: $\Omega = [0,T] \times \mathcal{S}$, where

$$\mathcal{S} := \Big\{(b,s) \in \mathbb{R}^2 \Big|\ b+(1-\mu)s > 0,\ b+(1+\lambda)s > 0\Big\}.$$



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Also: In this case we have $V(t,b,s) = -\infty$ for $(b,s) \in \partial\mathcal{S}$.

Our Contribution

Uniqueness follows from the following **comparison principle**.

Comparison Principle

Let u and v be continuous functions from Ω to $\mathbb{R}$. Assume that for some $p \in (0,1)$, there exists $C > 0$ such that

$$|u(t,b,s)| \leq C(1 + |b| + |s|)^p, \qquad |v(t,b,s)| \leq C(1 + |b| + |s|)^p.$$

If u is a viscosity subsolution and v is a viscosity supersolution and if $u \leq v$ on $\partial\Omega$, then $u \leq v$ on Ω.

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Our contribution is to **remove the growth condition**.

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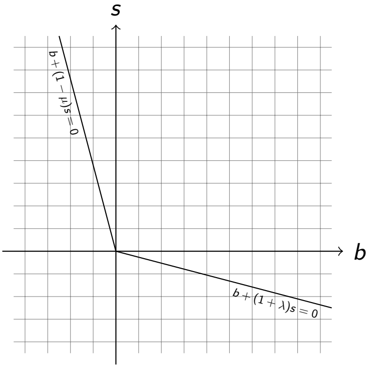
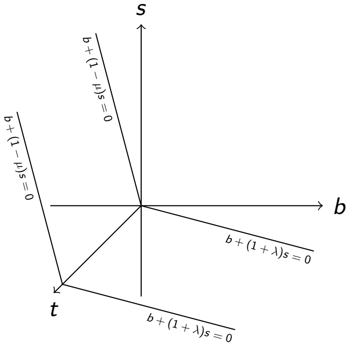
The Idea of the Proof





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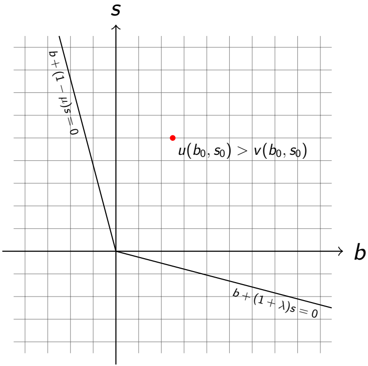

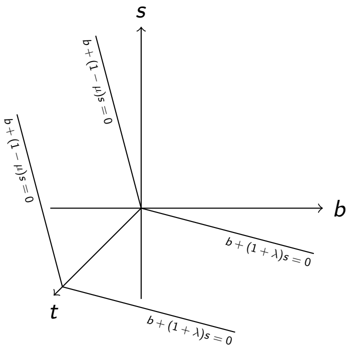

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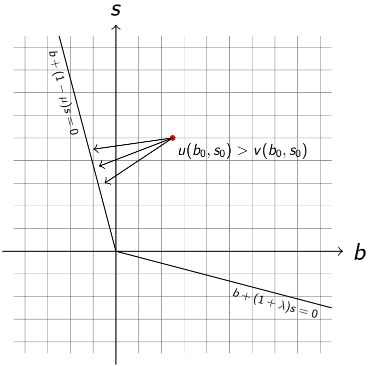

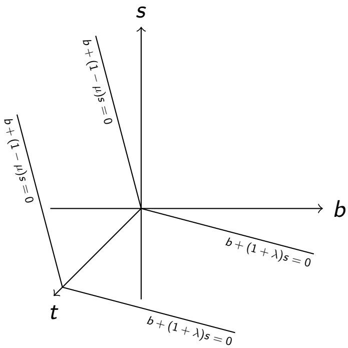

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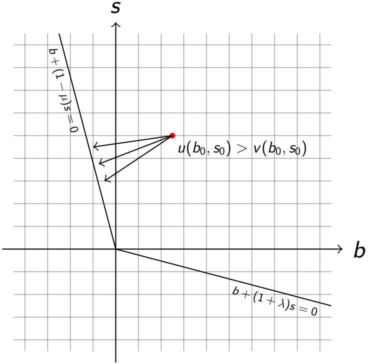

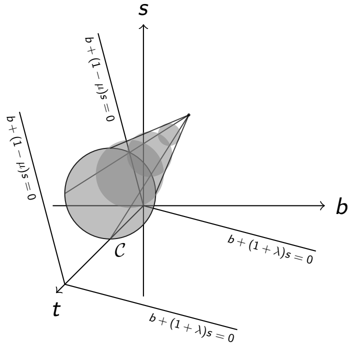

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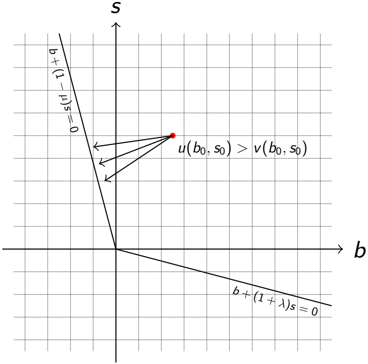

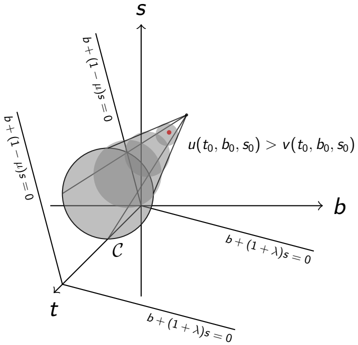

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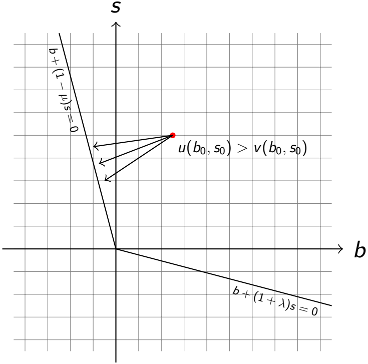

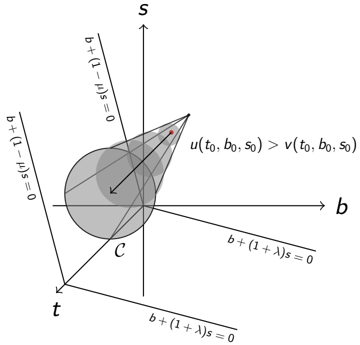

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Plugging u and v in the HJB equation, we obtain a contradiction as we let $n \to \infty$.

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3. The uniqueness result therefore shows that the function V is a **classical solution** of the HJB equation.
4. Furthermore, when we do **numerics**, we are now assured that we are simulating the correct function.

Thank you for your attention!!!

Full details in the Preprint:

C. Belak, O. Menkens, J. Sass: *Worst-Case Portfolio Optimization with Proportional Transaction Costs* (2013).

Literature

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