

Optimal Terminal Wealth with Transaction Costs and Uniqueness of Unbounded Viscosity Solutions

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Outline

- 1 Optimal Terminal Wealth with Transaction Costs
- 2 Viscosity Solutions of Second-Order PDEs
- 3 Uniqueness of unbounded Viscosity Solutions

The Market Model

We consider the following financial market. The investor can only invest in a risk-free asset (“**bond**”) or a risky asset (“**stock**”).

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Assume the wealth invested in the bond and stock, resp., follows

$$\begin{aligned} dB_t &= rB_{t-} dt & B_{0-} &= b, \\ dS_t &= \alpha S_{t-} dt + \sigma S_{t-} dW_t & S_{0-} &= s. \end{aligned}$$

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L_t : “**investment strategy**”, cumulative amount of money used for **buying** shares of the stock (increasing, càdlàg),

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The investor’s **net wealth** X is then

$$X_t = \begin{cases} B_t + (1 - \mu)S_t, & \text{if } S_t > 0, \\ B_t + (1 + \lambda)S_t, & \text{if } S_t \leq 0. \end{cases}$$

Admissibility and Solvency

We say that a trading strategy (L, M) is **admissible**, if the corresponding net wealth process X is a.s. nonnegative.

This is equivalent to saying that the pair (B_t, S_t) stays inside the closure of the following **solvency cone**:

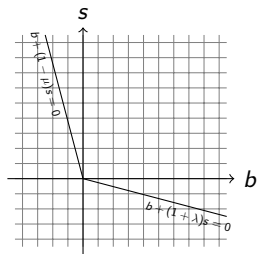
$$\mathcal{S} := \{(b, s) \in \mathbb{R}^2 : b + (1 - \mu)s > 0, b + (1 + \lambda)s > 0\}.$$

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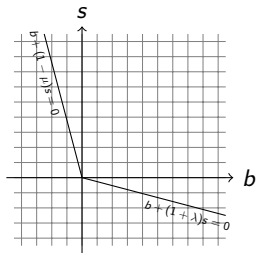


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For a given initial position $(b, s) \in \mathcal{S}$, we denote the corresponding set of **admissible strategies** by $\mathcal{A}(b, s)$.

Problem Formulation

We consider the following **control problem**:

$$V(t, b, s) = \sup_{(L, M) \in \mathcal{A}(b, s)} \mathbb{E}_{t, b, s} [U_p(X_T)],$$

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where $U_p : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a **utility function** of the form

$$U_p(x) = \begin{cases} \frac{1}{p} x^p, & \text{if } p < 1, p \neq 0, \\ \log(x), & \text{if } p = 0. \end{cases}$$

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This problem has been studied by many authors:

Magill/Constantinides (1976),
 Davis/Norman (1990), Shreve/Soner (1994),
 Akian/Séquier/Sulem (1995), Dai/Yi (2009), Bichuch (2012).

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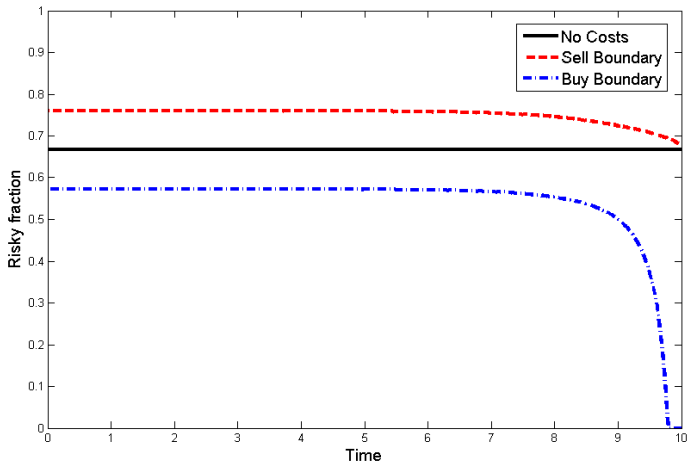
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In total, this means h has to solve the **HJB equation**

$$0 = \max \left\{ \mathcal{L}^{nt} h(t, b, s), \mathcal{L}^{buy} h(t, b, s), \mathcal{L}^{sell} h(t, b, s) \right\}.$$

Numerical Example: Optimal Trading Regions



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- Uniqueness allows to treat the problem **numerically**.
- The existence of the optimal strategies is a **work in progress**.

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The idea behind Viscosity Solutions

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Viscosity Solutions: A Formal Definition

Assume that F is continuous and decreasing in its last argument. Let Ω be an open set and let w be continuous on Ω . Then

- 1 w is called a **viscosity subsolution** of $F = 0$, if for all $x_0 \in \Omega$ and all $\varphi \in C^2(\Omega)$ such that $w - \varphi$ has a local maximum at x_0 , we have

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It turns out that the notion of viscosity solutions is the **perfect concept** for solutions of HJB-equations!

Application to our Transaction Cost Problem

Consider the **HJB equation**

$$0 = \min\{\mathcal{L}^{nt} h(t, b, s), \mathcal{L}^{buy} h(t, b, s), \mathcal{L}^{sell} h(t, b, s)\}, \quad [0, T) \times \mathcal{S},$$

where the differential operators are given by

$$\mathcal{L}^{nt} h = -h_t - rbh_b - \alpha sh_s - \frac{1}{2} \sigma^2 s^2 h_{ss},$$

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Then the value function V is a **viscosity solution** with boundary conditions

$$V(t, b, s) = U(0+), \quad \text{on } [0, T) \times \partial\mathcal{S},$$

$$V(T, b, s) = \begin{cases} U(b + (1 - \mu)s), & \text{if } s > 0, \\ U(b + (1 + \lambda)s), & \text{if } s \leq 0. \end{cases} \quad \text{on } \{T\} \times \mathcal{S}.$$

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Therefore, if the HJB equation admits at most one viscosity solution, then the value function must also be a classical solution.

Outline

- 1 Optimal Terminal Wealth with Transaction Costs
- 2 Viscosity Solutions of Second-Order PDEs
- 3 Uniqueness of unbounded Viscosity Solutions

The Comparison Principle

In order to show uniqueness, one typically proves a **comparison principle**.

Comparison Principle

Let u be a viscosity subsolution and let v be a viscosity supersolution. Suppose that there exist constants $C > 0$ and $p \in (0, 1)$ such that

$$|u(t, x)| \leq C(1 + |x|)^p, \quad |v(t, x)| \leq C(1 + |x|)^p.$$

If $u \leq v$ on the boundary, then $u \leq v$ on $[0, T] \times \bar{S}$.

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Problem: The growth condition ensures that the value function does not grow too rapidly when x becomes large. If the value function mainly depends on the choice of the utility function, i.e.

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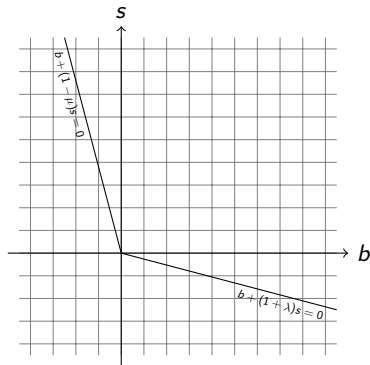
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The real problem is to control the value function **near the boundary!**

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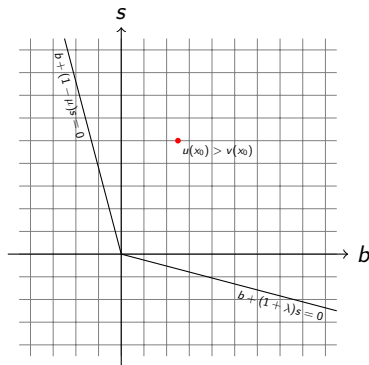
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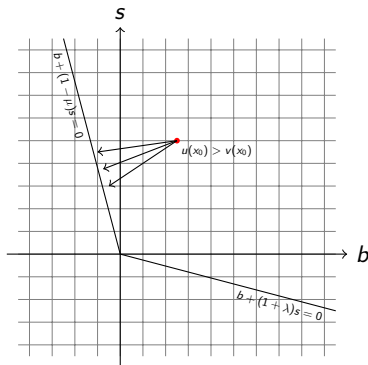
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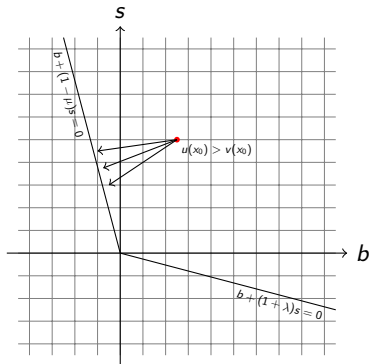
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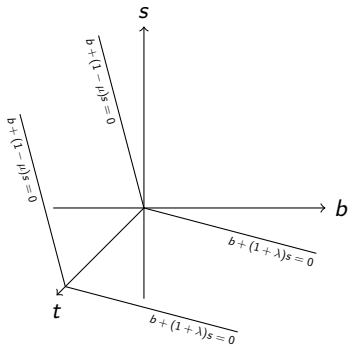
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However, we cannot do this if we do not know how fast u and v grow.



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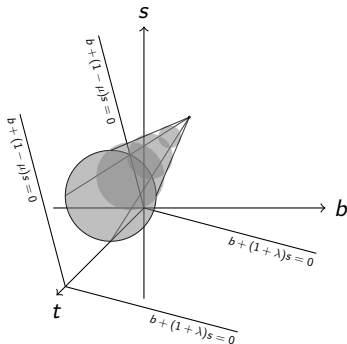
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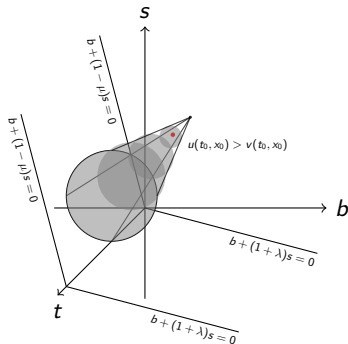
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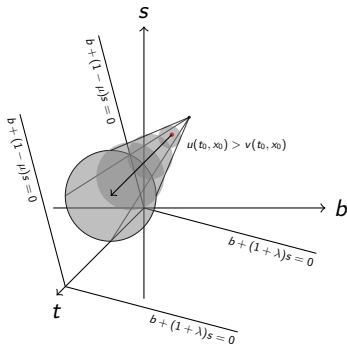
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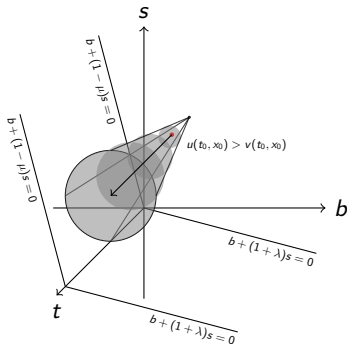
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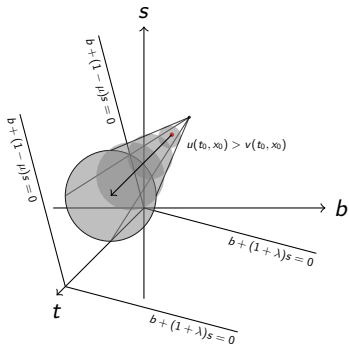
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Since the cone is chosen arbitrarily, the comparison theorem holds everywhere.



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- 5 Next step: Show that the optimal controls exist.

Thank you for your attention!!!