Worst-Case Portfolio Optimization

with Proportional Transaction Costs

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Motivation

Worst-Case Portfolio Optimization with Transaction Costs

- We study **optimal asset allocation** in a simple financial market.
- For every transaction, the investor has to **pay a fee** proportional to the size of the transaction.
- We assume that the market is under the **threat of a crash**.

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- We study **optimal asset allocation** in a simple financial market.
- For every transaction, the investor has to **pay a fee** proportional to the size of the transaction.
- We assume that the market is under the **threat of a crash**.

For every trading strategy, we identify the **worst-case scenario** in terms of market crashes and look for the strategy which **performs best** if the worst-case scenario is realized.

Worst-Case Portfolio Problem

$$\sup_{\pi \in \mathcal{A}, \bar{\pi} \in \bar{\mathcal{A}}} \inf_{\tau \in \mathcal{B}} E[U(X_T)].$$

Market Model and Trading Strategies

Trading in a Market with Proportional Transaction Costs

Assume the wealth invested in the bond and stock, resp., follows

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\begin{aligned}
dB_t &= rB_{t-}dt \\
dS_t &= \alpha S_{t-}dt + \sigma S_{t-}dW_t
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Market Model and Trading Strategies

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Assume the wealth invested in the bond and stock, resp., follows

$$dB_t = rB_{t-}dt - (1+\lambda)dL_t$$
$$dS_t = \alpha S_{t-}dt + \sigma S_{t-}dW_t + dL_t$$

L_t : "**investment strategy**", represents the cumulative amount of money used for buying stock (increasing, càdlàg),
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The **net wealth** of an investor is given by

$$X(t) = \begin{cases} B(t) + (1-\mu)S(t), & \text{if } S(t) > 0, \\ B(t) + (1+\lambda)S(t), & \text{if } S(t) \leq 0. \end{cases}$$

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4. For simplicity: $\beta_\tau \equiv \beta \in (0, 1)$ **constant**. This is no restriction, since we take a worst-case perspective!
5. We assume that the investor can observe crashes and **adjust the trading strategy** afterward. To be more precise, the investor chooses a pre-crash strategy (L, M) and a family of post-crash strategies $\{(L^\tau, M^\tau)\}$.

Problem Formulation

Worst-Case Terminal Wealth Problem

$$\mathcal{V}(t,b,s) = \sup_{\mathcal{A}(t,b,s),\bar{\mathcal{A}}(L,M)} \inf_{\mathcal{B}(L,M)} E_{t,b,s}[U_p(X_T)].$$

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- $\mathcal{B}(L,M)$ denotes the set of all **crash times** with respect to $\mathbb{F}^{L,M}$.
- U_p is a **utility function** of the form

$$U_p(x) = \begin{cases} \frac{1}{p}x^p, & \text{if } p<1, p\neq 0,\\ \log x, & \text{if } p=0. \end{cases}$$

The Dynamic Programming Principle

Denote by

$$\bar{\mathcal{V}}(t,b,s) = \sup_{\mathcal{A}(t,b,s)} E_{t,b,s}[U_p(X_T)]$$

be the value function in the corresponding crash-free market. Then we have the following result.

Dynamic Programming Principle

Let θ be a $[t,T]$-valued stopping time. Then

$$\mathcal{V}(t,b,s) = \sup_{\mathcal{A}(t,b,s)} \inf_{\mathcal{B}(L,M)} E_{t,b,s}\Big[\mathcal{V}(\theta, B_{\theta-}, S_{\theta-})\mathbf{1}_{\{\theta<\tau\}} + \bar{\mathcal{V}}(\tau, B_{\tau-}, (1-\beta)S_{\tau-})\mathbf{1}_{\{\tau\leq\theta\}}\Big].$$

An immediate consequence of the dynamic programming principle is the following **crash constraint**:

$$\mathcal{V}(t,b,s) \leq \bar{\mathcal{V}}(t,b,(1-\beta)s).$$

The Dynamic Programming Equation (crash-free market)

It is a well-known result that $\bar{\mathcal{V}}$ is a **viscosity solution** of

$$0 = \min\left\{\mathcal{L}^{nt}\bar{\mathcal{V}}(t,b,s), \mathcal{L}^{buy}\bar{\mathcal{V}}(t,b,s), \mathcal{L}^{sell}\bar{\mathcal{V}}(t,b,s)\right\},$$

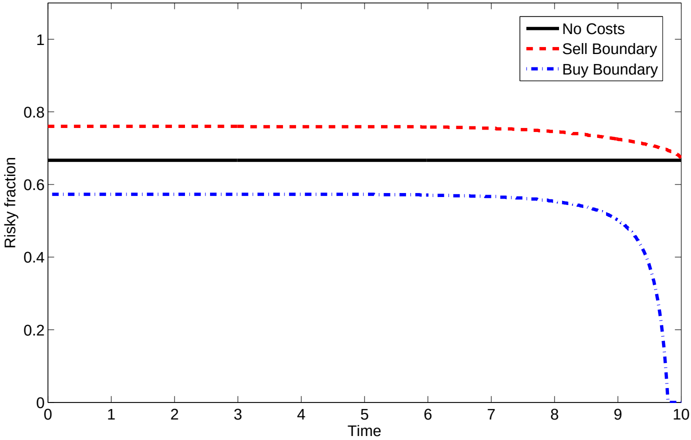
where

$$\begin{aligned}
\mathcal{L}^{nt}\bar{\mathcal{V}} &= -\bar{\mathcal{V}}_t - \alpha s\bar{\mathcal{V}}_s - rb\bar{\mathcal{V}}_b - \frac{1}{2}\sigma^2 s^2\bar{\mathcal{V}}_{ss}, \\
\mathcal{L}^{buy}\bar{\mathcal{V}} &= (1+\lambda)\bar{\mathcal{V}}_b - \bar{\mathcal{V}}_s, \\
\mathcal{L}^{sell}\bar{\mathcal{V}} &= -(1-\mu)\bar{\mathcal{V}}_b + \bar{\mathcal{V}}_s.
\end{aligned}$$

If $0 < p < 1$, then $\bar{\mathcal{V}}$ is known to be **unique** in a suitable class of functions.

The operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ determine the **optimal action** of the investor.

Numerical Results: Zero Crashes



The Dynamic Programming Equation (crash market)

$\mathcal{V}$ is a **viscosity solution** of

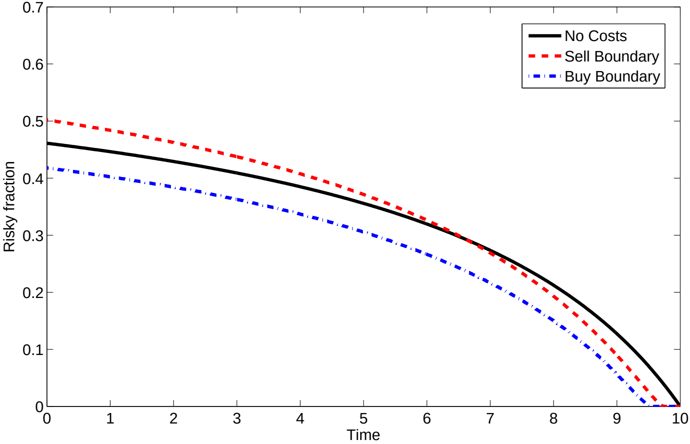
$$0 = \max\Big\{\mathcal{V}(t,b,s) - \bar{\mathcal{V}}(t,b,(1-\beta)s),$$
$$\min\big\{\mathcal{L}^{nt}\mathcal{V}(t,b,s), \mathcal{L}^{buy}\mathcal{V}(t,b,s), \mathcal{L}^{sell}\mathcal{V}(t,b,s)\big\}\Big\}.$$

If $0 < p < 1$, then $\mathcal{V}$ is **unique** in a suitable class of functions.

A **crash** (in the worst-case sense) can only occur if the **crash constraint** holds with equality:

$$\mathcal{V}(t,b,s) = \bar{\mathcal{V}}(t,b,(1-\beta)s).$$

Numerical Results: One Crash, 20%



Thank you for your attention!!!

Example

Assume that the investor has a **positive stock position** s at time $T-\varepsilon$ and assume that **at most one crash** can still occur. Let $r=0$.
A crash at time $T-\varepsilon$ would result in

$$S_{T-\varepsilon} = (1-\beta)s.$$

If ε is sufficiently small, the stock position evolves approximately like a **geometric Brownian motion**. Therefore

$$E[S_T] \approx (1-\beta)se^{\alpha\varepsilon}.$$

A requirement for this strategy to be optimal is that

$$(1-\beta)se^{\alpha\varepsilon} \geq s \qquad \Leftrightarrow \qquad (1-\beta)e^{\alpha\varepsilon} \geq 1,$$

since otherwise the **pure bond strategy** performs better.