Worst-Case Portfolio Optimization

with Proportional Transaction Costs

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Motivation

Worst-Case Portfolio Optimization with Transaction Costs

- We study **optimal asset allocation** in a simple financial market.
- We assume that the market is under the **threat of a crash**.
- For every transaction, the investor has to **pay a fee** proportional to the size of the transaction.

Our aim is to find a strategy, which

- **maximizes** wealth at terminal time $T > 0$ and
- **protects** the investor against the occurrence of a crash.

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$$dS_t = \alpha S_{t-}dt + \sigma S_{t-}dW_t + dL_t$$

L_t : "**investment strategy**", cumulative amount of money used for buying stock (increasing, càdlàg),
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$\lambda, \mu \in (0,1)$: "**transaction costs**".

N_t: "**crash process**", counting the number of crashes,
$\beta \in (0,1)$: "**crash height**".

Properties of the Crash Process N_t

1. $N(t)$ counts the **number of crashes** up to time t, i.e.

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5. $\beta \in (0, 1)$ can be seen as an **upper bound** for the crash height.
6. The maximal number of crashes in $[0, T]$ is assumed to be **bounded**.
7. We assume that the investor can observe crashes and **adjust the trading strategy** afterwards.

Admissibility of Trading Strategies

We are interested in maximizing the **net wealth after liquidation** of the stock position:

$$X(t) = \begin{cases} B(t) + (1-\mu)S(t), & \text{if } S(t) > 0, \\ B(t) + (1+\lambda)S(t), & \text{if } S(t) \leq 0. \end{cases}$$

This leads to the following definition of a **solvency region**:

$$\mathcal{S}^n = \{(b,s) \in \mathbb{R}^2 : b + (1+\lambda)s > 0, b + (1-\mu)(1-\beta)^n s > 0\}.$$

A trading strategy (L, M) is called **admissible** if

$$\left(B^{L,M}(u), S^{L,M}(u)\right) \in \overline{\mathcal{S}^n}, \qquad \text{for all } u \in [t, T],$$

whenever there are at most n crashes left up to maturity T.

Problem Formulation

Our objective is to solve the following **optimization problem**.

Worst-Case Terminal Wealth Problem

$$\mathcal{V}^n(t,b,s) = \sup_{(L,M)} \inf_{N} E_{t,b,s,n}[U_p(X_T)]$$

We assume that U_p is a **utility function** of the form

$$U_p(x) = \begin{cases} \frac{1}{p}x^p, & \text{if } p < 1, p \neq 0, \\ \log x, & \text{if } p = 0. \end{cases}$$

The Dynamic Programming Principle

Our main tool in the analysis of the optimization problem is the **dynamic programming principle**.

Dynamic Programming Principle

Let θ be a $[t, T]$-valued stopping time. Then

$$\mathcal{V}^n(t, b, s) = \sup_{L,M} \inf_{\tau} E_{t,b,s,n}\Big[\mathcal{V}^n(\theta, B_{\theta-}, S_{\theta-})\mathbf{1}_{\{\theta<\tau\}} + \mathcal{V}^{n-1}(\tau, B_{\tau-}, (1-\beta)S_{\tau-})\mathbf{1}_{\{\tau\le\theta\}}\Big].$$

The infimum here is taken over all possible **first crash times** τ.

An immediate consequence of the dynamic programming principle is the following **crash constraint**:

$$\mathcal{V}^n(t, b, s) \le \mathcal{V}^{n-1}(t, b, (1-\beta)s).$$

The Dynamic Programming Equation for $n = 0$

If $n = 0$, then $\mathcal{V}^0$ is a **viscosity solution** of

$$0 = \min\big\{\mathcal{L}^{nt}\mathcal{V}^0(t,b,s), \mathcal{L}^{buy}\mathcal{V}^0(t,b,s), \mathcal{L}^{sell}\mathcal{V}^0(t,b,s)\big\},$$

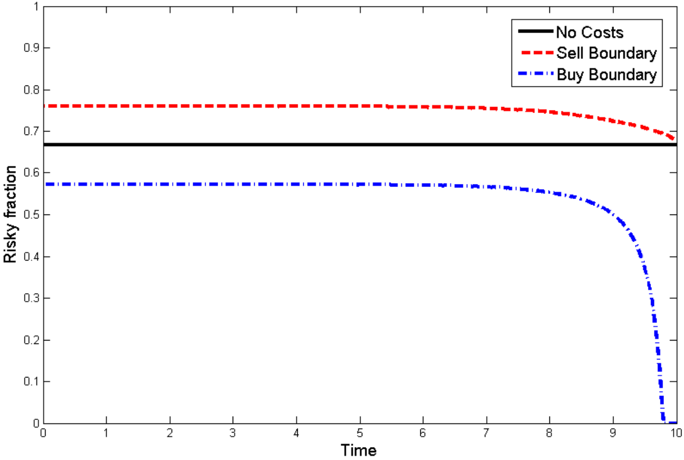
where

$$\begin{aligned}\mathcal{L}^{nt}\mathcal{V}^0 &= -\mathcal{V}^0_t - \alpha s\mathcal{V}^0_s - rb\mathcal{V}^0_b - \frac{1}{2}\sigma^2 s^2\mathcal{V}^0_{ss},\\ \mathcal{L}^{buy}\mathcal{V}^0 &= (1+\lambda)\mathcal{V}^0_b - \mathcal{V}^0_s,\\ \mathcal{L}^{sell}\mathcal{V}^0 &= -(1-\mu)\mathcal{V}^0_b + \mathcal{V}^0_s.\end{aligned}$$

If $0 < p < 1$, then $\mathcal{V}^0$ is known to be **unique** in a suitable class of functions.

The operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ determine the **optimal action** of the investor.

Numerical Results: Zero Crashes



The Dynamic Programming Equation for $n > 0$

If $n > 0$, then $\mathcal{V}^n$ is a **viscosity solution** of

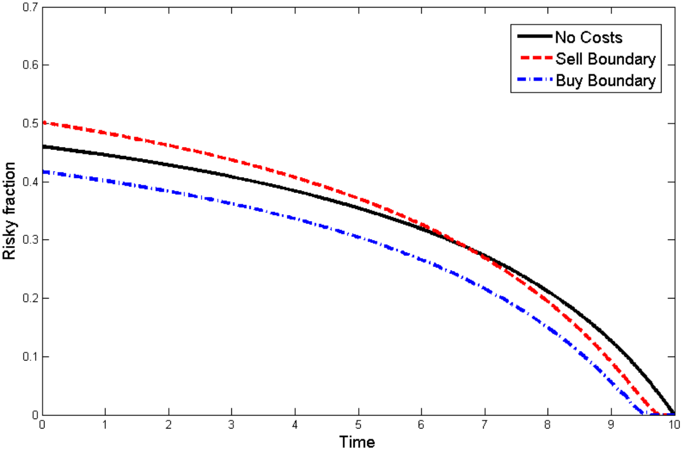
$$0 = \max\Big\{\mathcal{V}^n(t,b,s) - \mathcal{V}^{n-1}(t,b,(1-\beta)s),\\ \min\big\{\mathcal{L}^{nt}\mathcal{V}^n(t,b,s), \mathcal{L}^{buy}\mathcal{V}^n(t,b,s), \mathcal{L}^{sell}\mathcal{V}^n(t,b,s)\big\}\Big\}.$$

If $0 < p < 1$, then $\mathcal{V}^n$ is **unique** in a suitable class of functions.

A **crash** (in the worst-case sense) can only occur if the **crash constraint** holds with equality:

$$\mathcal{V}^n(t,b,s) = \mathcal{V}^{n-1}(t,b,(1-\beta)s).$$

Numerical Results: One Crash, 20%



Example

Assume that the investor has a **positive stock position** s at time $T-\varepsilon$ and assume that **at most one crash** can still occur. Let $r=0$.
A crash at time $T-\varepsilon$ would result in

$$S_{T-\varepsilon}=(1-\beta)s.$$

If ε is sufficiently small, the stock position evolves approximately like a **geometric Brownian motion**. Therefore

$$E[S_T]\approx(1-\beta)se^{\alpha\varepsilon}.$$

A requirement for this strategy to be optimal is that

$$(1-\beta)se^{\alpha\varepsilon}\geq s \qquad\Leftrightarrow\qquad (1-\beta)e^{\alpha\varepsilon}\geq 1,$$

since otherwise the **pure bond strategy** performs better.

Thank you for your attention!!!