Worst-Case Portfolio Optimization

with Proportional Transaction Costs

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Joint work with

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Motivation

Worst-Case Portfolio Optimization with Transaction Costs

- We study **optimal asset allocation** in a simple financial market.
- We assume that the market is under the **threat of a crash**.
- For every transaction, the investor has to **pay a fee** proportional to the size of the transaction.

Our aim is to find a strategy, which

- **maximizes** wealth at terminal time $T > 0$ and
- **protects** the investor against the occurrence of a crash.

The Market Model

We consider the following financial market. The investor can only invest in a risk-free asset ("**bond**") or a risky asset ("**stock**").

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Assume the wealth invested in the bond and stock, resp., follows

$$dB_t = rB_{t-}dt$$
$$dS_t = \alpha S_{t-}dt + \sigma S_{t-}dW_t$$

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$$dB_t = rB_{t-}dt - (1+\lambda)dL_t$$
$$dS_t = \alpha S_{t-}dt + \sigma S_{t-}dW_t + dL_t$$

L_t : "**investment strategy**", cumulative amount of money used for buying stock (increasing, càdlàg),
$\lambda \in (0,1)$: "**transaction costs**".

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$$dS_t = \alpha S_{t-}dt + \sigma S_{t-}dW_t + dL_t - dM_t - \beta S_{t-}dN_t.$$

L_t, M_t: "**investment strategy**", cumulative amount of money used for buying and selling stock (increasing, càdlàg),
$\lambda, \mu \in (0,1)$: "**transaction costs**".

N_t: "**crash process**", counting the number of crashes (bounded),
$\beta \in (0,1)$: "**crash height**".

Problem Formulation

The investor's **wealth after liquidation of the stock position** is

$$X_t = \begin{cases} B_t + (1-\mu)S_t, & \text{if } S_t > 0, \\ B_t + (1+\lambda)S_t, & \text{if } S_t \leq 0. \end{cases}$$

Our objective is to solve the **optimization problem**

$$\mathcal{V}^n(t,b,s) = \sup_{(L,M)\in\mathcal{A}} \inf_{N\in\mathcal{B}} E_{t,b,s,n}[U(X_T)],$$

where U is the **power utility function**:

$$U(x) = \frac{1}{p}x^p, \qquad 0 < p < 1.$$

The Dynamic Programming Principle

Our main tool in the analysis of the optimization problem is the **dynamic programming principle**.

Dynamic Programming Principle for Crash Times

For every admissible crash time $\tilde{\tau}$, set $\tau := (\tilde{\tau}-) \wedge T$. Then the **dynamic programming principle** holds:

$$\mathcal{V}^n(t,b,s) = \sup_{L,M} \inf_{\tau} E\Big[\mathcal{V}^{n-1}(\tau, B_\tau, (1-\beta)S_\tau)\Big].$$

An immediate consequence of the dynamic programming principle is the following **crash constraint**:

$$\mathcal{V}^n(t,b,s) \leq \mathcal{V}^{n-1}(t,b,(1-\beta)s).$$

The Dynamic Programming Equation for $n = 0$

If $n = 0$, then $\mathcal{V}^0$ is the **unique viscosity solution** of

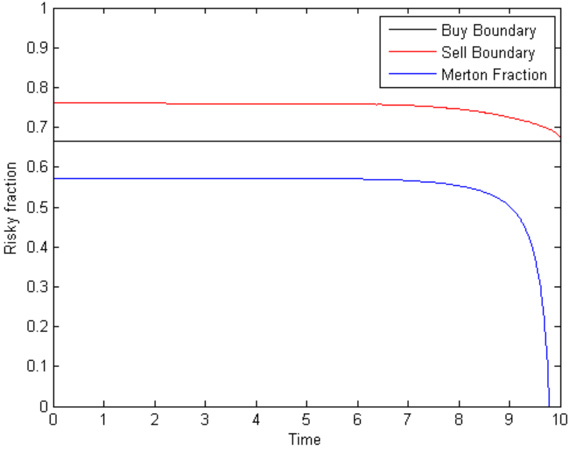
$$0 = \min\left\{\mathcal{L}^{nt}\mathcal{V}^0(t,b,s), \mathcal{L}^{buy}\mathcal{V}^0(t,b,s), \mathcal{L}^{sell}\mathcal{V}^0(t,b,s)\right\},$$

where

$$\begin{aligned}
\mathcal{L}^{nt}\mathcal{V}^0 &= -\mathcal{V}^0_t - \alpha s\mathcal{V}^0_s - rb\mathcal{V}^0_b - \frac{1}{2}\sigma^2 s^2 \mathcal{V}^0_{ss}, \\
\mathcal{L}^{buy}\mathcal{V}^0 &= (1+\lambda)\mathcal{V}^0_b - \mathcal{V}^0_s, \\
\mathcal{L}^{sell}\mathcal{V}^0 &= -(1-\mu)\mathcal{V}^0_b + \mathcal{V}^0_s.
\end{aligned}$$

Depending on which operator vanishes determines the **optimal action** of the investor.

Numerical Results: Zero Crashes



The Dynamic Programming Equation for $n > 0$

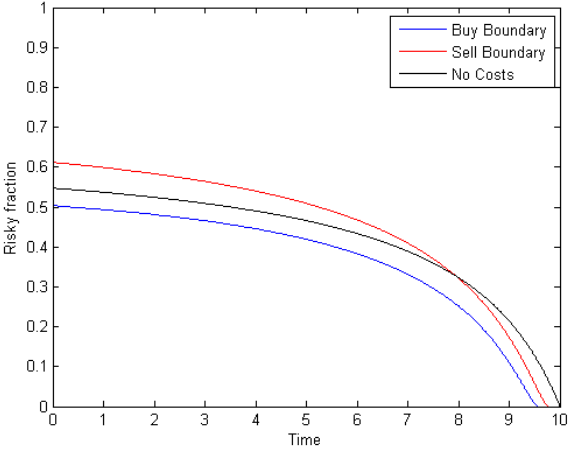
If $n > 0$, then $\mathcal{V}^n$ is the **unique viscosity solution** of

$$0 = \max\Big\{\mathcal{V}^n(t,b,s) - \mathcal{V}^{n-1}(t,b,(1-\beta)s), \\ \min\big\{\mathcal{L}^{nt}\mathcal{V}^n(t,b,s), \mathcal{L}^{buy}\mathcal{V}^n(t,b,s), \mathcal{L}^{sell}\mathcal{V}^n(t,b,s)\big\}\Big\}.$$

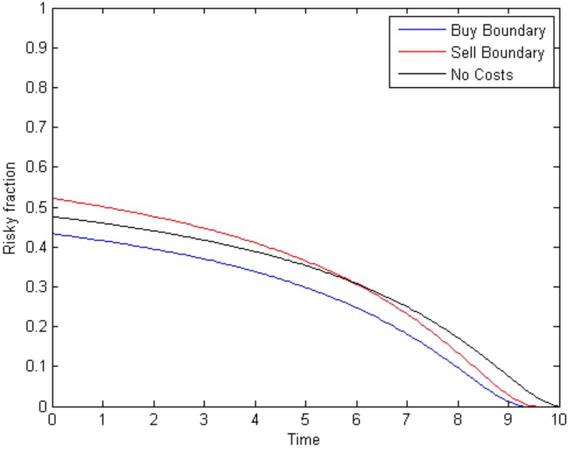
A **crash** (in the worst-case sense) can only occur if the **crash constraint** holds with equality:

$$\mathcal{V}^n(t,b,s) = \mathcal{V}^{n-1}(t,b,(1-\beta)s).$$

Numerical Results: One Crash, 10%



Numerical Results: Two Crashes, 10%



Example

Assume that the investor has a **positive stock position** at time $T - \varepsilon$ and assume that **at most one crash** can still occur. Let $r = 0$.
A crash at time $T - \varepsilon$ would result in

$$S_{T-\varepsilon} = (1 - \beta)s.$$

If ε is sufficiently small, the stock position evolves approximately like a **geometric Brownian motion**. Therefore

$$E[S_T] \approx (1 - \beta)se^{\alpha\varepsilon}.$$

A requirement for this strategy to be optimal is that

$$(1 - \beta)se^{\alpha\varepsilon} \geq s \qquad \Leftrightarrow \qquad (1 - \beta)e^{\alpha\varepsilon} \geq 1,$$

since otherwise the **pure bond strategy** performs better.

Thank you for your attention!!!