Optimal Investment: Transaction Costs and Crash Threats

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Overview

Market Model: The Black-Scholes Market

We consider a **financial market** consisting of two tradable assets:

1. A **risk-free asset** (bond or bank account) with dynamics

$$dB_t = rB_t\,dt.$$

2. A **risk-bearing asset** (stock) with dynamics

$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t.$$

The **total wealth** X_t of an investor then evolves according to

$$dX_t = (1-\eta_t)X_t r\,dt + \eta_t X_t(\alpha\,dt + \sigma\,dW_t), \qquad X_0 = 1,$$

where η_t denotes the fraction of the total wealth invested in the risky asset at time t, the so-called **risky-fraction process**.

The Optimal Investment Problem

The objective is to find an investment strategy η_t^* which yields the highest **expected utility of the total wealth** at some terminal time T. That is, we face the problem

$$\sup_{\eta_t \in \mathcal{A}} E\left[U_p\left(X_T\right)\right].$$

We assume that U_p is a **utility function** of the form

$$U_p(x) = \begin{cases} \frac{1}{p}x^p & \text{if } p < 1, p \neq 0, \\ \log x & \text{if } p = 0. \end{cases}$$

The problem was solved in 1969 by Robert C. **Merton**.

The Log-Utility Case

We consider the log-utility case $U_0(x) = \log x$. The wealth process

$$dX_t = (1 - \eta_t) X_t r \, dt + \eta_t X_t (\alpha \, dt + \sigma \, dW_t), \qquad X_0 = 1,$$

can be written as

$$X_t = \exp \left\{ \int_0^t (1 - \eta_u) r + \eta_u \alpha - \frac{1}{2} \sigma^2 \eta_u^2 \, du + \int_0^t \sigma \eta_u \, dW_u \right\}.$$

Thus the problem reduces to

$$\sup_{\eta_t} E \left[U_0 \left(X_T \right) \right] = \sup_{\eta_t} E \left[\int_0^T (1 - \eta_u) r + \eta_u \alpha - \frac{1}{2} \sigma^2 \eta_u^2 \, du \right].$$

Pointwise optimization yields

$$\eta_t^* = \frac{\alpha - r}{\sigma^2}.$$

"Merton Fraction"

Shortcomings of the Merton Model

The model suggested by Merton has certain **shortcomings**. Possible extensions are:

1. The optimal solution in the Merton model requires the investor to continuously adjust his stock and bond positions. This does not work in practice, since transactions are usually subject to a fee.
   $\Rightarrow$ Include **transaction costs** into the model.

2. The stock price process is modelled as a continuous geometric Brownian motion which cannot describe extreme price movements.
   $\Rightarrow$ Include **crash scenarios** into the model.

Including Transaction Costs in the Model

Different Types of Transaction Costs

In the literature, one typically distinguishes between three different types of transaction costs:

1. **Constant Costs**: Pay constant C units of money per transaction.
2. **Fixed Costs**: Pay $c \cdot X_t$ for some fraction c of the total wealth at time t.
3. **Proportional Costs**: Pay $\gamma \cdot |\Delta|$ for some constant γ and transaction size Δ.

Constant and fixed costs punish the **frequency of trading** whereas proportional costs punish the **size of the transaction**.

The Market with Proportional Transaction Costs

Proportional Transaction Costs Models:
Davis/Norman (1990), Shreve/Soner (1994) and Day/Yi (2009).

Bond and Stock Dynamics

The amounts of money invested in bond and stock follow

$$dB_t = rB_t\,dt$$
$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t$$

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Bond and Stock Dynamics

The amounts of money invested in bond and stock follow

$$dB_t = rB_t \, dt - (1+\lambda) \, dL_t$$
$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t + dL_t$$

L_t: adapted, increasing, cadlag process representing the cumulative amount of money used for buying stock.

$\lambda \in (0, \infty)$: transaction costs for buying

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Bond and Stock Dynamics

The amounts of money invested in bond and stock follow

$$dB_t = rB_t\,dt - (1+\lambda)\,dL_t + (1-\mu)\,dM_t,$$
$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t + dL_t - dM_t.$$

L_t: adapted, increasing, cadlag process representing the cumulative amount of money used for buying stock.
M_t: adapted, increasing, cadlag process representing the cumulative amount of money obtained from selling stock.

$\lambda \in (0,\infty)$, $\mu \in (0,1)$: transaction costs for buying and selling.

Admissible Trading Strategies

Problem: At terminal time T, what to do with the stock position?

$$X_t = \begin{cases} B_t + (1-\mu)S_t & \text{if } S_t \geq 0, \\ B_t + (1+\lambda)S_t & \text{if } S_t < 0. \end{cases} \qquad \text{vs.} \qquad X_t = B_t + S_t.$$

This inspires the definition of the following **solvency region**:

$$\mathcal{S} := \left\{(b,s) \in \mathbb{R}^2 : b + (1+\lambda)s > 0, b + (1-\mu)s > 0\right\}$$

The set of **admissible strategies** is then given by

$$\mathcal{A}(t,b,s) = \left\{(L,M) : X_u^{L,M} \in \overline{\mathcal{S}} \text{ for all } u \in [t,T]\right\}.$$

Problem Formulation

For initial positions $(b, s) \in \overline{\mathcal{S}}$, the **optimal terminal wealth** problem then reads

$$\sup_{(L,M)\in\mathcal{A}(0,b,s)} E\left[U_p\left(X_T^{L,M}\right)\middle| B_0 = b, S_0 = s\right].$$

For $(t, b, s) \in [0, T) \times \overline{\mathcal{S}}$, we define the **value function** to be

$$V(t, b, s) = \sup_{(L,M)\in\mathcal{A}(t,b,s)} E\left[U_p\left(X_T^{L,M}\right)\middle| B_t = b, S_t = s\right].$$

If we want to be able to solve the optimization problem we need to assume that the value function is **finite**.

Dynamic Programming and the HJB-equation

In order to solve the portfolio problem, we use the theory of **stochastic optimal control**. One needs the so-called **dynamic programming principle** which states

$$V(t,b,s) = \sup_{(L,M)} E\left[V(\tau, B_\tau, S_\tau) | B_t = b, S_t = s\right].$$

for every stopping stime τ taking values in $[t,T)$. This can be used to show that the value function is the unique solution to the partial differential equation (**HJB-equation**)

$$0 = \min\Big\{-V_t - \alpha s V_s - rbV_b - \frac{1}{2}s^2\sigma^2 V_{ss}, \\ -(1-\mu)V_b + V_s, (1+\lambda)V_b - V_s\Big\}.$$

How to make these arguments rigorous?

How can we make these arguments rigorous?

1. Show that the value function satisfies the HJB-equation in a weak sense (**viscosity solution**).
2. Proof that the value function is sufficiently regular. This implies that the value function is also a **classical solution** of the HJB-equation.
3. Show that the optimal risky fraction process is a **diffusion process reflected at the boundaries** of $\mathcal{R}_t^{buy}$ and $\mathcal{R}_t^{sell}$.

Especially the regularity is hard to obtain.

A parabolic double obstacle problem

One can show that the HJB-equation is equivalent to the following **parabolic double obstacle problem**:

$$\begin{cases} -v_t - \mathcal{L}v = 0 & \text{if } \frac{1}{x+1+\lambda} < v(x,t) < \frac{1}{x+1-\mu} \\ -v_t - \mathcal{L}v \geq 0 & \text{if } v(x,t) = \frac{1}{x+1+\lambda} \\ -v_t - \mathcal{L}v \leq 0 & \text{if } v(x,t) = \frac{1}{x+1-\mu} \\ v(x,T) = \frac{1}{x+1-\mu} & \end{cases}$$

for some differential operator $\mathcal{L}$.

Including Crash Scenarios in the Model

Modelling Crashes of the Stock

There are different approaches to **modelling stock crashes**. E.g.

1. Replace the Brownian motion which drives the stock price with a **Lévy process**.

   $\Rightarrow$ Jump intensities are hard to estimate in practice.

2. Model the crash process as a **control variable** held by the market and optimize over all worst-case scenarios.

   $\Rightarrow$ Stochastic game between investor and market.

In the sequel, we will take a look at the second approach.

The Crash Market Model

Worst-Case Optimization:

Korn/Willmott (2002), Menkens (2004), Korn/Steffensen (2006), Seifried (2010).

Bond and Stock Dynamics

The amounts of money invested in bond and stock follow

$$dB_t = rB_t\,dt - (1+\lambda)\,dL_t + (1-\mu)\,dM_t,$$
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$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t + dL_t - dM_t - \beta S_t\,dN_t.$$

N_t: Crash process counting the number of crashes.

$\beta \in (0,1)$: crash height.

The Crash Process

Properties of the crash process N_t:

1. N_t **counts the number of crashes** up to time t.
2. We do not assume any distribution of the process. Instead, regard N_t as a control variable held by the market used to **minimize the total wealth** of the investor.
3. Thus, the problem can be regarded as a **game between the investor and the market**: The investor wants to find a strategy (L_t, M_t) in order to maximize expected utility of terminal wealth. The market wants to find a strategy N_t which minimizes this quantity.
4. We consider a finite horizon problem on $[0, T]$ and **limit the maximal number of crashes** in this interval to N_{max}.

Problem Formulation

The **optimization problem** in this model reads as follows

$$\sup_{(L,M)} \inf_{N} E\left[U_p\left(X_T^{L,M,N} \right) \middle| B_0 = b, S_0 = s \right].$$

The **value functions** in this case are given by

$$V^n(t,b,s) = \sup_{(L,M)} \inf_{N} E_{t,b,s,n}\left[U_p\left(X_T^{L,M,N} \right) \right].$$

Here $E_{t,b,s,n}$ denotes the conditional expectation given $B_t = b$, $S_t = s$ and that there are at most n crashes left.

Characterization of the Value Function

We have a result similar to the dynamic programming principle.

Theorem

The value function satisfies

$$\begin{aligned} V^n(t,b,s) &= \sup_{(L,M)} \inf_{N} E_{t,b,s,n}\left[U\left(X_T^{L,M,N}\right)\right] \\ &= \sup_{(L,M)} \inf_{\tau} E_{t,b,s,n}\left[V^{n-1}\left(\tau, \hat{B}_\tau, \hat{S}_\tau\right)\right], \end{aligned}$$

where

$$\begin{aligned} \hat{B}_t &= B_{t-} - (1+\lambda)(L_t - L_{t-}) + (1-\mu)(M_t - M_{t-}), \\ \hat{S}_t &= (1-\beta)S_{t-} + (L_t - L_{t-}) - (M_t - M_{t-}). \end{aligned}$$

Optimal Crash Time

The last theorem already allows us to deduce the **optimal crash strategy**.

Theorem

Assume that at time t the market has n crash options left. Then the optimal crash time τ_n^* is given by

$$\tau_n^* = \inf \left\{ u \geq t : V^n \left(u, B_u, S_u \right) \geq V^{n-1} \left(u, \hat{B}_u, \hat{S}_u \right) \right\}.$$

Optimal Trading with Crashes

In a crash-threatened market one can show, that in between crash times **the HJB-equation still holds**. However, we obtain an additional constraint to become **indifferent about crashes**.

Theorem

The value function satisfies the HJB equation

$$0 = \min\Big\{ -V_t^n - \alpha s V_s^n - rbV_b^n - \frac{1}{2}\sigma^2 s^2 V_{ss}^n, \\ -(1-\mu)V_b^n + V_s^n, (1+\lambda)V_b^n - V_s^n \Big\}$$

under the constraint

$$V^n(t,b,s) \leq V^{n-1}(t,b,(1-\beta)s).$$

Any Questions??

Thank you for your attention!!